Spectrum Analysis

Periodic Signals

$$v(t) = \sum_{n=-\infty}^{\infty} v_r(t - nT_r)$$

Periodic Signals

Since v(t) is repetitive we can represent v(t) as a sum of orthogonal repetitive functions. We will choose sine waves

$$e^{j 2\pi m t/T_r} = \cos\left(2\pi m \frac{t}{T_r}\right) + j \sin\left(2\pi m \frac{t}{T_r}\right)$$

is periodic at T_r given m is an integer. Let

$$\omega_r = \frac{2\pi}{T_r}$$

Fourier Series

Then we can write the signal as

$$\boxed{v(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_r t}}$$

where C_m is complex. Multiply both sides by $e^{-jk\omega_r t}$ and integrate over a period.

$$\boxed{\int_{-T_r/2}^{T_r/2} v(t)\, e^{-jk\omega_r t}\, dt = \sum_{m=-\infty}^{\infty} C_m \int_{-T_r/2}^{T_r/2} e^{j(m-k)\omega_r t} = C_m T_r \delta_{m,k}}$$

Fourier Series

$$\boxed{C_m = \frac{1}{T_r}\int_{-T_r/2}^{T_r/2} v(t)\, e^{-jm\omega_r t}\, dt}$$

Note that since v (t) is real $\boxed{C_{-m} = C_m^*}$

Let $\boxed{\omega_m = m\,\omega_r}$

$$\boxed{\tilde{V}(\omega_m) = T_r\, C_m}$$

Units of Volts/ Hz

Fourier Transforms

$$v(t) = \sum_{m=-\infty}^{\infty} \frac{1}{T_r} \tilde{V}(\omega_m)\, e^{j\omega_m t}$$

The spacing between adjacent frequencies is:

$$\Delta\omega_m = m\frac{2\pi}{T_r} - (m-1)\frac{2\pi}{T_r} = \frac{2\pi}{T_r}$$

The Fourier series becomes:

$$v(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \tilde{V}(\omega_m)\, e^{j\omega_m t}\, \Delta\omega_m$$

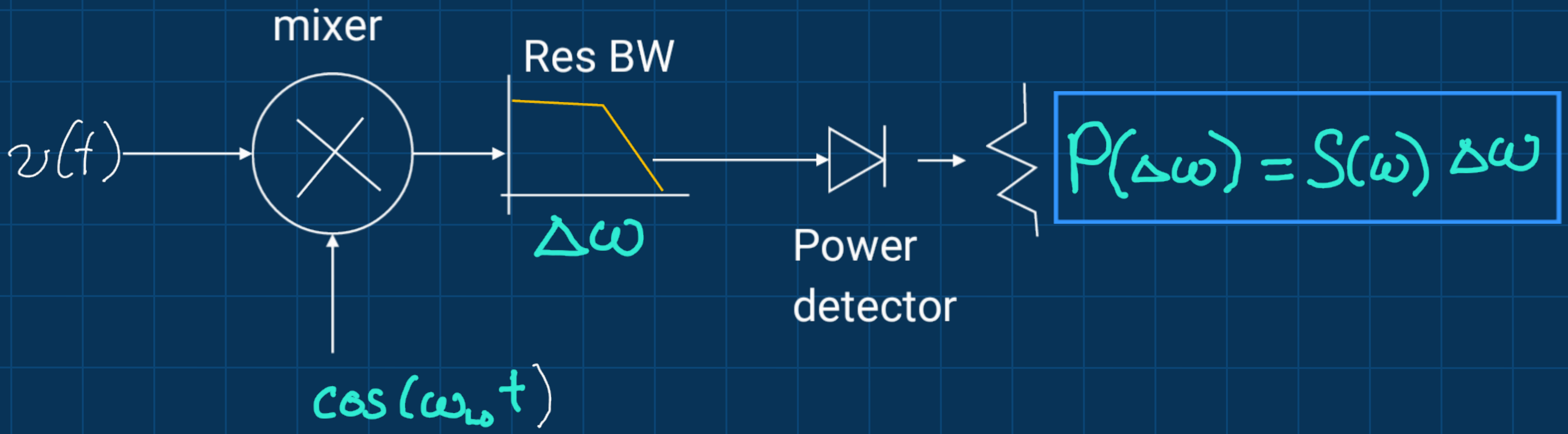
Fourier Transforms

For non-periodic signals, take the limit as $T_1 \to \infty \quad (\Delta\omega_m \to 0)$

$$\boxed{\begin{aligned} v(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}(\omega)\, e^{j\omega t}\, d\omega \\ \tilde{V}(\omega) &= \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

Spectrum Analysis

Swept spectrum analyzers do not measure voltages and currents. They measure power deposited into a filter of width given by the resolution bandwidth.



Power Spectral Density

The power spectral density is defined as:

$$\boxed{\langle p(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega}$$

But time averaged power is:

$$\boxed{\langle p(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \frac{1}{R} \int_{-T/2}^{T/2} v(t)\, v(t)\, dt}$$

Power Spectral Density

Use the fact that because v(t) is real:

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}(\omega_1) e^{j\omega_1 t}\, d\omega_1$$

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}(\omega_2)^{*} e^{-j\omega_2 t}\, d\omega_2$$

and

$$\delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\, d\omega$$

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\, dt$$

results

$$\langle p(t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \frac{|V(\omega)|^2}{R}\, d\omega$$

Power Spectral Density

The power spectral density reduces to:

$$\boxed{S_f(\omega) = \lim_{T \to \infty} \frac{1}{T} \frac{1}{R} |\tilde{V}(\omega)|^2}$$

Since $\tilde{V}(\omega)$ has units of Volts/Hz,

$S_f(\omega)$ has units of Watts/ Hz

Power Spectral Density

Consider the periodic signal again

$$V_p(t) = \sum_{n=-\infty}^{\infty} V_r(t - nT_r)$$

Written as a Fourier Series

$$V_p(t) = \sum_{m=-\infty}^{\infty} C_m e^{j m \omega_r t}$$

Taking the Fourier transform

$$\tilde{V}_p(\omega) = \sum_{m=-\infty}^{\infty} C_m \int_{-\infty}^{\infty} e^{j(m\omega_r - \omega)\tau} \, d\tau$$

Results with:

$$\tilde{V}_p(\omega) = 2\pi \sum_{m=-\infty}^{\infty} C_m \, \delta(\omega - m\omega_r)$$

Power Spectral Density

Using the magic of delta functions with the limit as T goes to infinity, the gifted reader can show:

$$\boxed{S_p(\omega) = \frac{2\pi}{R} \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_r)}$$

Since

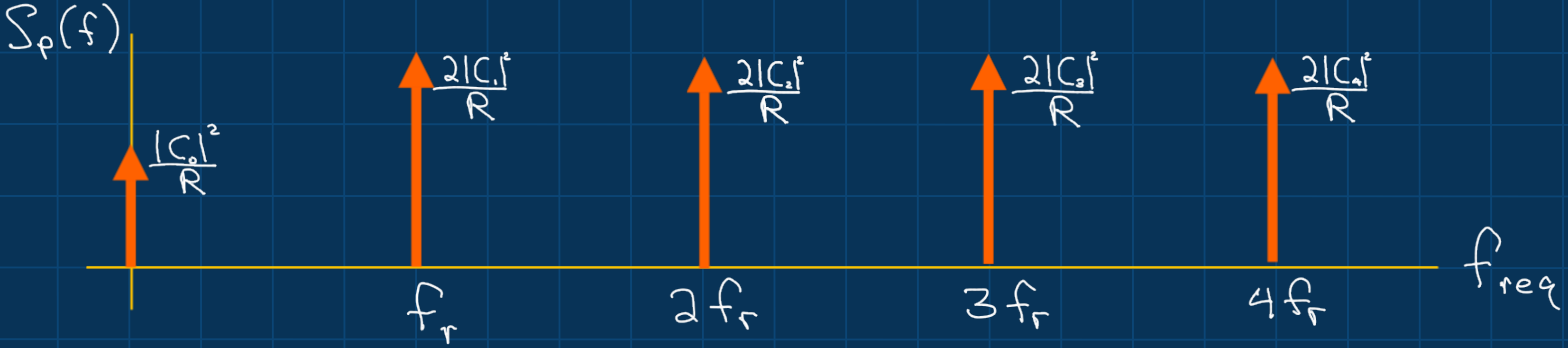
$$\boxed{\omega = 2\pi f}$$

$$\boxed{S_p(f) = \frac{1}{R} \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - n f_r)}$$

Spectrum Analysis

Since spectrum analyzers do not measure phase, they do not display negative frequencies

$$\boxed{S_p(f) = \frac{1}{R}|C_0|^2\delta(0) + \frac{2}{R}\sum_{m=1}^{\infty}|C_m|^2\delta(f-nf_r)}$$



Spectrum Analysis

- Note that a spectrum analyzer does not measure the power spectral density.
- It measures the power deposited in a filter of width called the resolution bandwidth at a frequency f.
- So <u>for a periodic signal</u>, as the resolution bandwidth is changed, the peak signal on a spectrum analyzer <u>does not vary</u>.