Introduction to RF for Particle Accelerators
Part 3: Beam Signals

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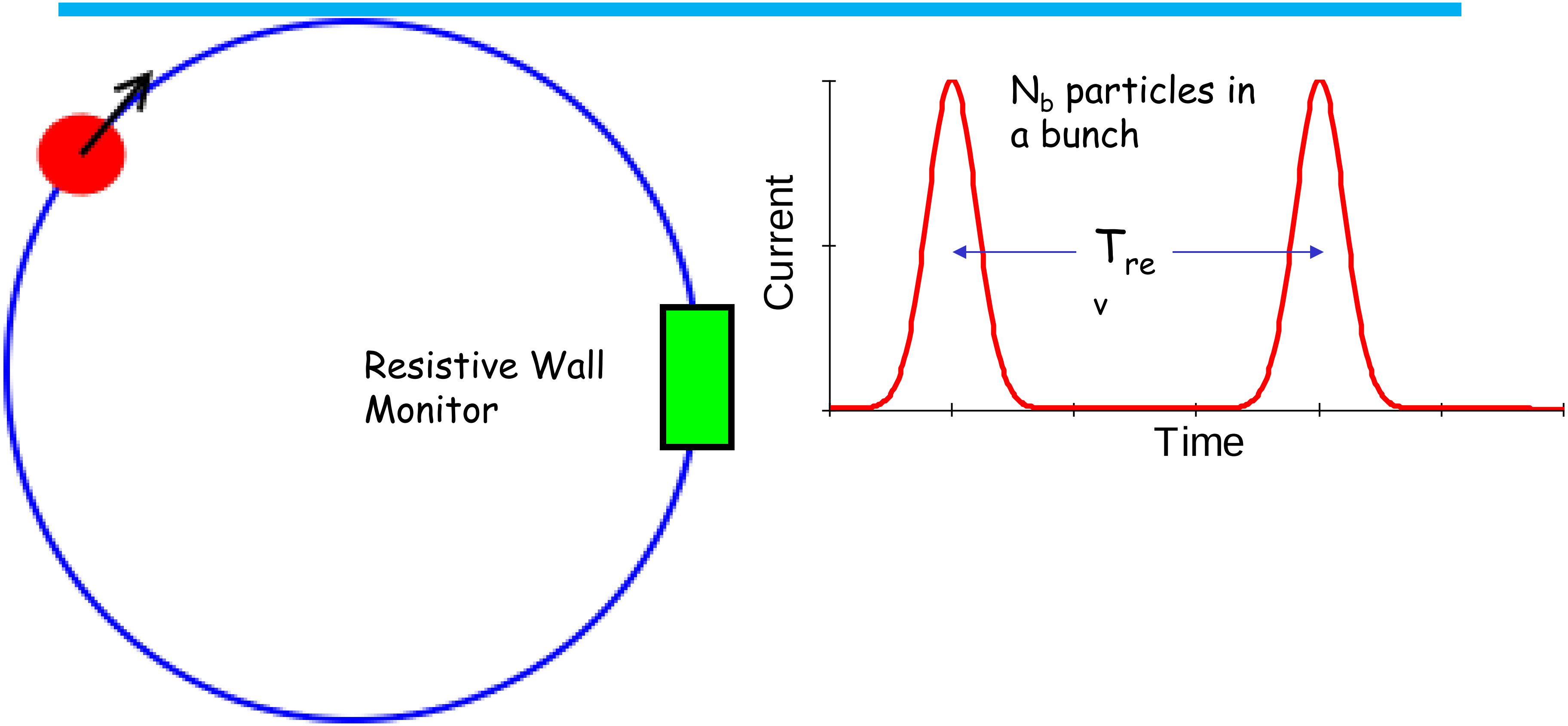
Beam Signals Outline

- Single Bunch in a Circular Ring
  - Fourier Series
  - Fourier Transforms
  - Delta functions
  - Power Spectral Density
  - Bunch Length Monitor
- M Equally Spaced Bunches in a Ring
- A Burst of Bunches in a Ring
- Betatron Motion
  - AM Modulation
- Longitudinal Motion
  - Bessel Function Magic
  - Frequency Modulation
- Multipole Distributions

Single Bunch in a Circular Ring



Bunch shape = f(t) where $\int f(t)dt = 1$

$$i_b(t) = \sum_{n=-\infty}^{\infty} qN_b f(t - nT_{rev})$$

Fourier Series

$$i_b(t) = \sum_{n=-\infty}^{\infty} qN_b f(t - nT_{rev})$$

This is a periodic series which can be expanded in a Fourier series

$$i_b(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_{rev} t}$$

where

$$\omega_{rev} = \frac{2\pi}{T_{rev}}$$

Fourier Series

$$C_m = \frac{qN_b}{T_{rev}} \int_{-T_{rev}/2}^{T_{rev}/2} f(\tau) e^{-jm\omega_{rev}\tau} d\tau$$

Note:

C_o is the DC beam Current

$$C_0 = \frac{qN_b}{T_{rev}}$$

f(t) is a real function

$$C_m = C_m^*$$

Fourier Transforms

Fourier Transform

$$i(t) = \int_{-\infty}^{\infty} \tilde{I}(2\pi f) e^{j2\pi f t} df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform

$$\tilde{I}(\omega) = \int_{-\infty}^{\infty} i(\tau) e^{-j\omega\tau} d\tau$$

Detour on Delta functions

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}(\omega) e^{j\omega t} d\omega$$

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{-\infty} i(\tau) e^{-j\omega\tau} d\tau \right) e^{j\omega t} d\omega$$

$$i(t) = \int_{-\infty}^{\infty} i(\tau) \left(\frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{j\omega(t-\tau)} d\omega \right) d\tau$$

$$\delta(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{j\omega(t-\tau)} d\omega$$

Fourier Transform of the Beam Current

$$\tilde{I}_b(\omega) = \int_{-\infty}^{\infty} i_b(\tau) e^{-j\omega\tau} d\tau$$

$$i_b(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_{rev} t}$$

$$\tilde{I}_b(\omega) = 2\pi \sum_{m=-\infty}^{\infty} C_m \delta(\omega - \omega_{rev})$$

Fourier Transform of the Beam Current

But spectrum analyzers do not measure currents and voltages.

They measure POWER deposited into a filter of width df

The total power into the spectrum analyzer

$$\langle p(t) \rangle = \int_{-\infty}^{\infty} S(2\pi f)\, df$$

$S(2\pi f)$ is the power spectral density and is the power measured by the spectrum analyzer in a resolution bandwidth of 1 Hz

Fourier Transform of the Beam Current

Time Averaged Power

$$\langle p(t) \rangle = \lim_{T\to\infty} \frac{1}{T} R \int_{-T/2}^{T/2} i(t) \cdot i(t) dt$$

Since:

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}(\omega) e^{j\omega t} d\omega$$

and i(t) is real

$$i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}^{*}(\omega) e^{-j\omega t} d\omega$$

Power Spectral Density

$$\langle p(t)\rangle = \lim_{T\to\infty}\frac{1}{T}R\int_{-T/2}^{T/2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{I}(\omega_1)e^{j\omega_1 t}d\omega_1\right)\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{I}^*(\omega_2)e^{-j\omega_2 t}d\omega_2\right)dt$$

Twizzle, Twazzle….

$$\langle p(t)\rangle = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}\left|\tilde{I}(\omega)\right|^2 R\frac{d\omega}{2\pi}$$

$$S(\omega) = \lim_{T\to\infty}\frac{1}{T}\left|\tilde{I}(\omega)\right|^2 R$$

Beam Power Spectrum

$$\tilde{I}_b(\omega) = 2\pi \sum_{m=-\infty}^{\infty} C_m \delta(\omega - \omega_{rev})$$

$$S_b(\omega) = \lim_{T\to\infty} \frac{R}{T}(2\pi)^2 \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} C_m C_{m'}^{*} \delta(\omega - m\omega_{rev})\delta(\omega - m'\omega_{rev})$$

Since delta functions do not overlap for $m \neq m'$

$$S_b(\omega) = \lim_{T\to\infty} \frac{R}{T}(2\pi)^2 \sum_{m=-\infty}^{\infty} |C_m|^2 (\delta(\omega - m\omega_{rev}))^2$$

Beam Power Spectrum

Some more delta function magic…

$$\lim_{T\to\infty}\frac{1}{T}\left(\delta(\omega-m\omega_{rev})\right)^2=\lim_{T\to\infty}\frac{1}{T}\delta(\omega-m\omega_{rev})\frac{1}{2\pi}\int_{-T/2}^{T/2}e^{j(\omega-m\omega_{rev})t}dt$$

$$=\frac{1}{2\pi}\delta(\omega-m\omega_{rev})$$

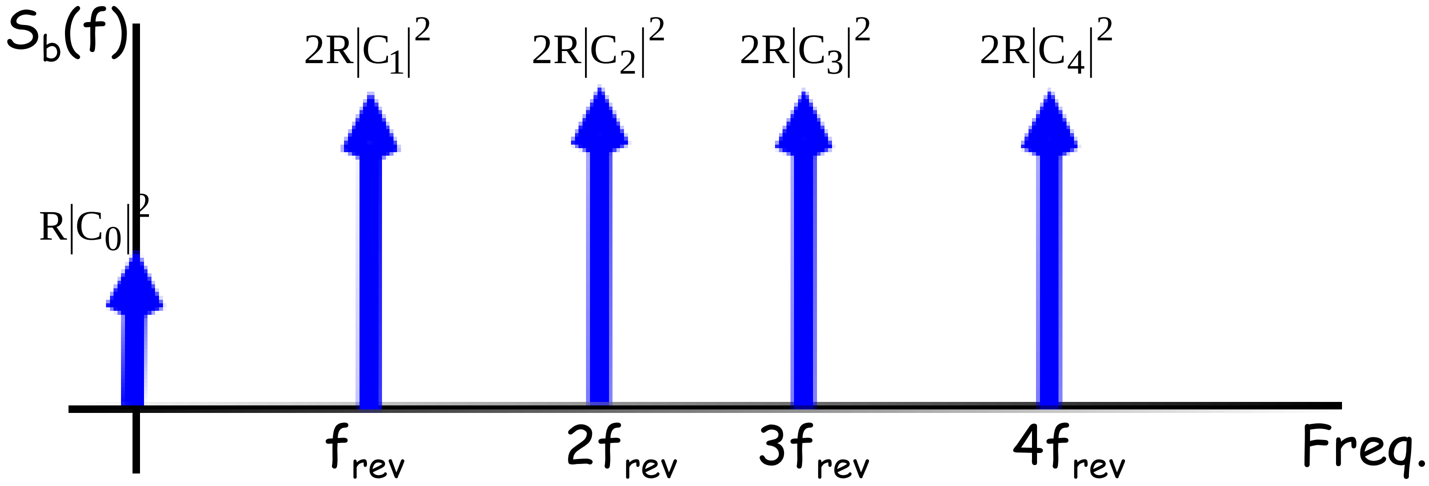
$$S_b(\omega)=2\pi R\sum_{m=-\infty}^{\infty}|C_m|^2\delta(\omega-m\omega_{rev})$$

$$S_b(f)=R\sum_{m=-\infty}^{\infty}|C_m|^2\delta(f-mf_{rev})$$

Beam Power Spectrum

Since spectrum analyzers can't distinguish between negative and positive frequencies:

$$S_b(f) = \left(\frac{qN_b}{T_{rev}}\right)^2 R\delta(0) + 2R\sum_{m=1}^{\infty}\left|C_m\right|^2 \delta(f - mf_{rev})$$



Beam Power Spectrum

The power contained in each revolution line:

$$P_0 = R\left(\frac{qN_b}{T_{rev}}\right)^2 \qquad m = 0$$

$$P_m = 2R\left|C_m\right|^2 \qquad m > 0$$

Note that if:

Short Bunch

$$f(t) = \delta(t)$$

$$P_m = 2R\left(\frac{qN_b}{T_{rev}}\right)^2$$

Long Bunch

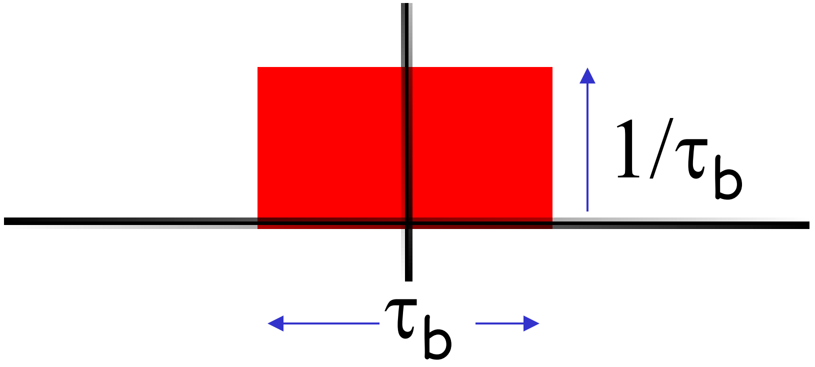
$$f(t) = \frac{1}{T_{rev}}\left(1 + \cos\left(2\pi f_{rev} t\right)\right)$$

$$P_1 = \frac{1}{2}R\left(\frac{qN_b}{T_{rev}}\right)^2$$

$$P_m = 0 \qquad m > 1$$

Bunch Length Monitor

Consider a square bunch with length τ_b



The power in spectral line m:

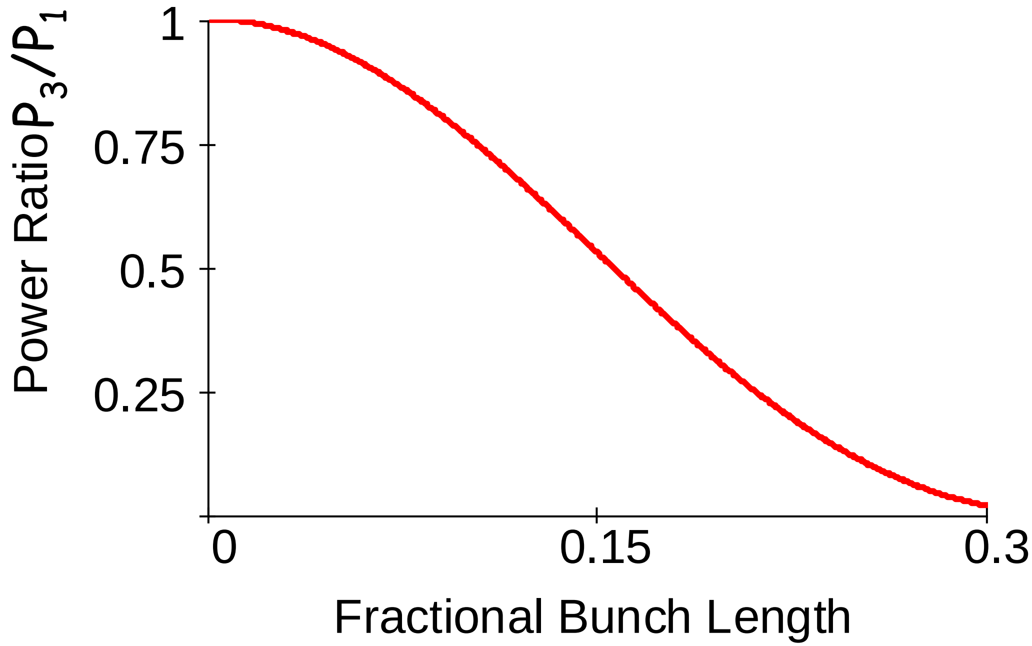
$$P_m = 2P_0 \mathrm{Sa}\left(m\pi \frac{\tau_b}{T_{rev}} \right)$$

$$\mathrm{Sa}(x) = \frac{\sin(x)}{x}$$

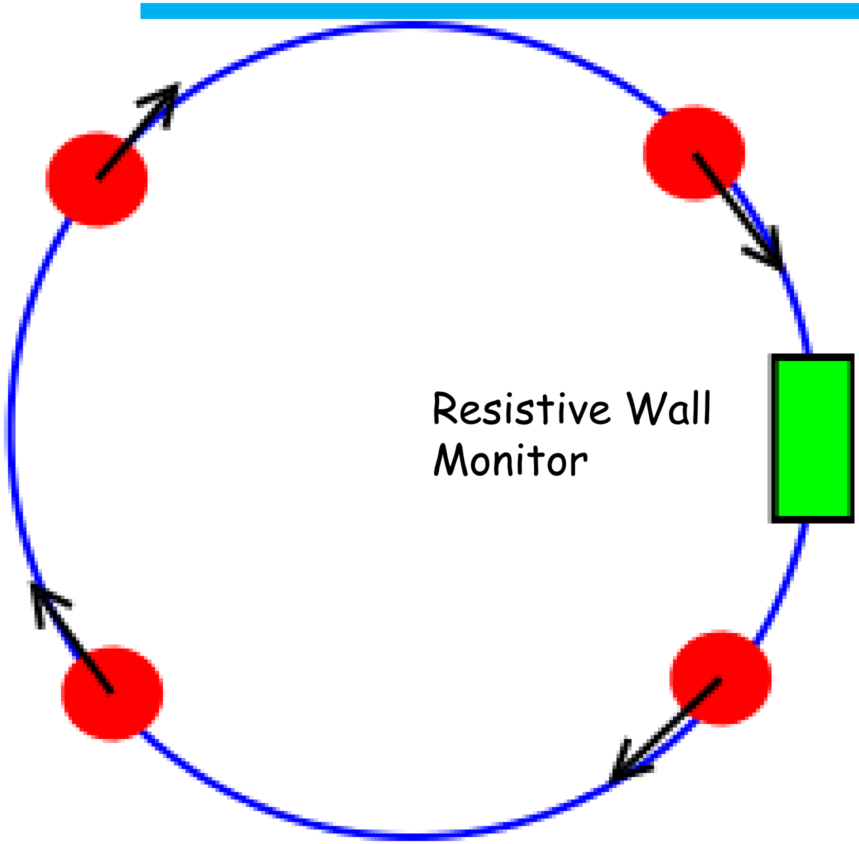
Bunch Length Monitor

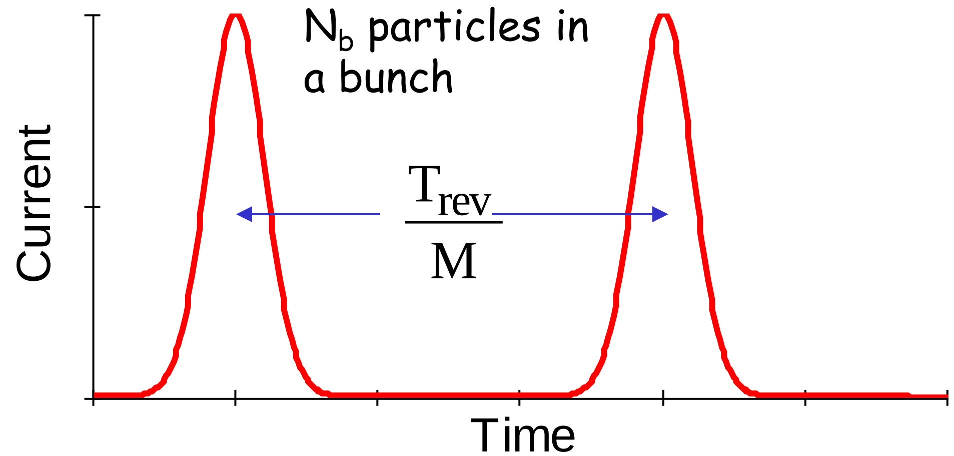
The ratio of the power in line n to the power in line m is independent of the beam intensity and is a function of the bunch length.

$$\frac{P_n}{P_m} = \frac{\mathrm{Sa}\left(n\pi\frac{\tau_b}{T_{rev}}\right)}{\mathrm{Sa}\left(m\pi\frac{\tau_b}{T_{rev}}\right)}$$



M Equally Spaced Bunches in a Ring





This looks exactly like 1 bunch in a machine M times smaller

$$i_b(t) = \sum_{n=-\infty}^{\infty} q N_b f\left(t - n\frac{T_{rev}}{M}\right)$$

M Equally Spaced Bunches in a Ring

$$i_b(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm(M\omega_{rev})t}$$

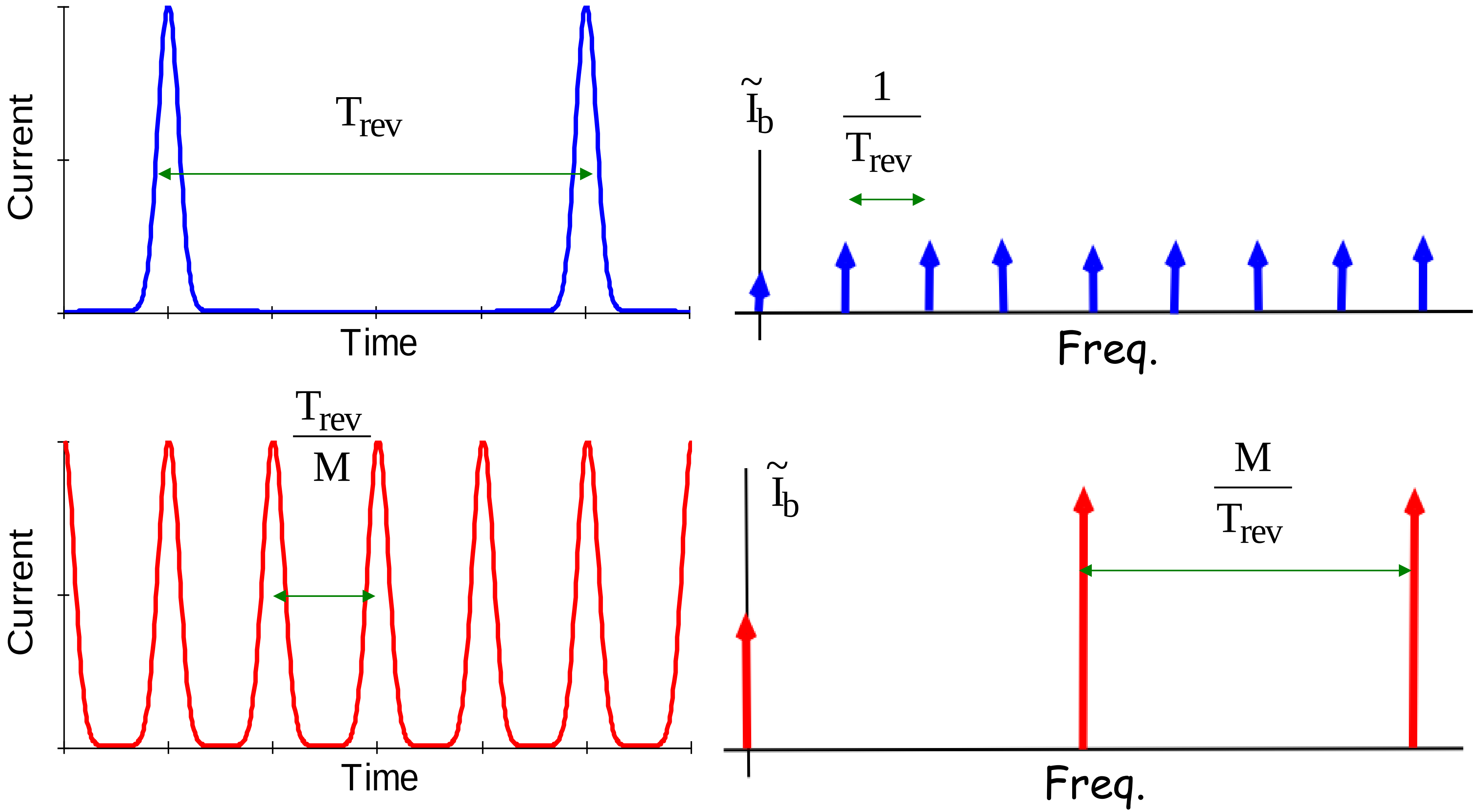
$$C_m = \frac{qN_b}{T_{rev}/M} \int_{-T_{rev}/2}^{T_{rev}/2} f(\tau) e^{-jm(M\omega_{rev})\tau} d\tau$$

Note that all the coefficients are M times bigger than for a single bunch. More bunches – more power.
For Example:

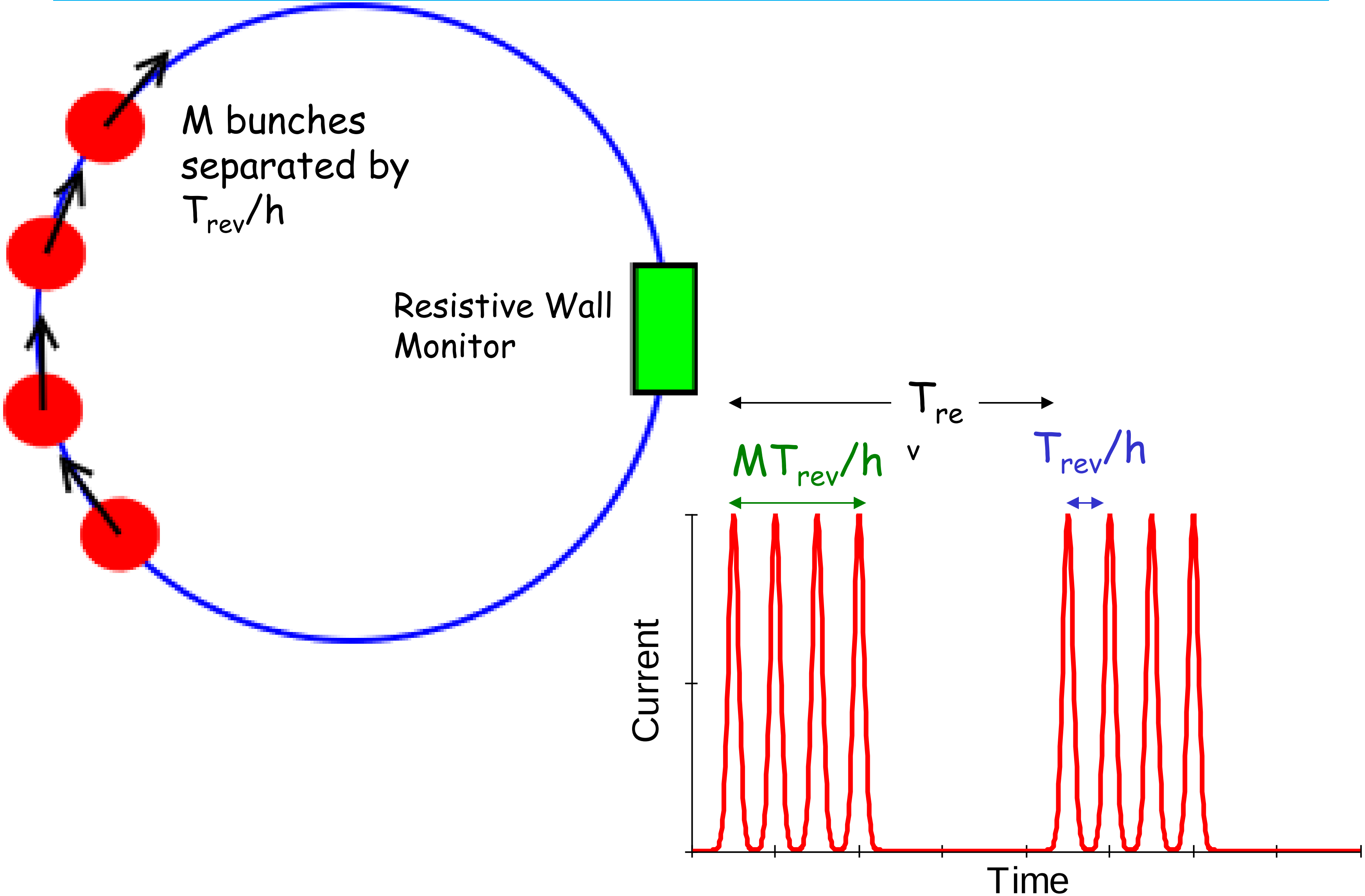
$$C_0 = M \frac{qN_b}{T_{rev}}$$

Which is still the total DC current in the ring

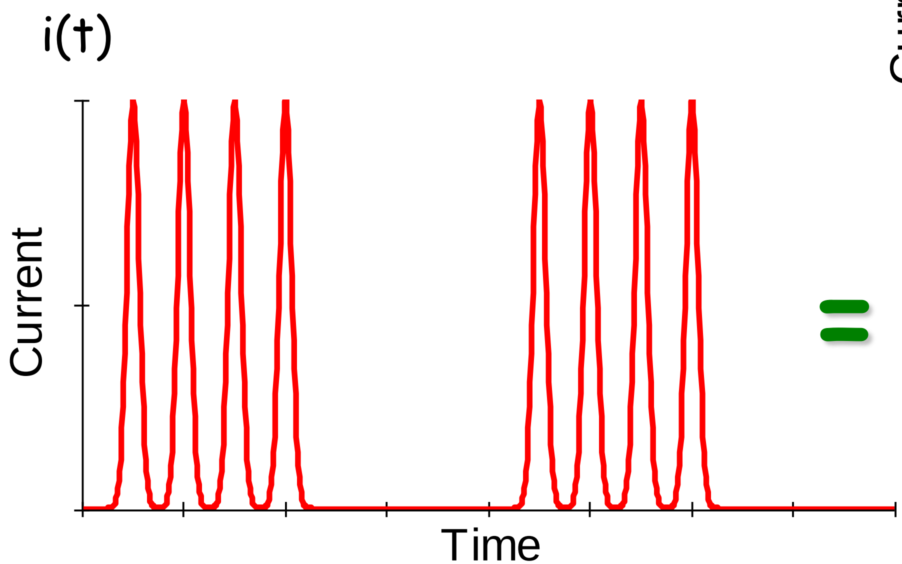
M Equally Spaced Bunches in a Ring



A Burst of Bunches in a Ring



A Burst of Bunches in a Ring



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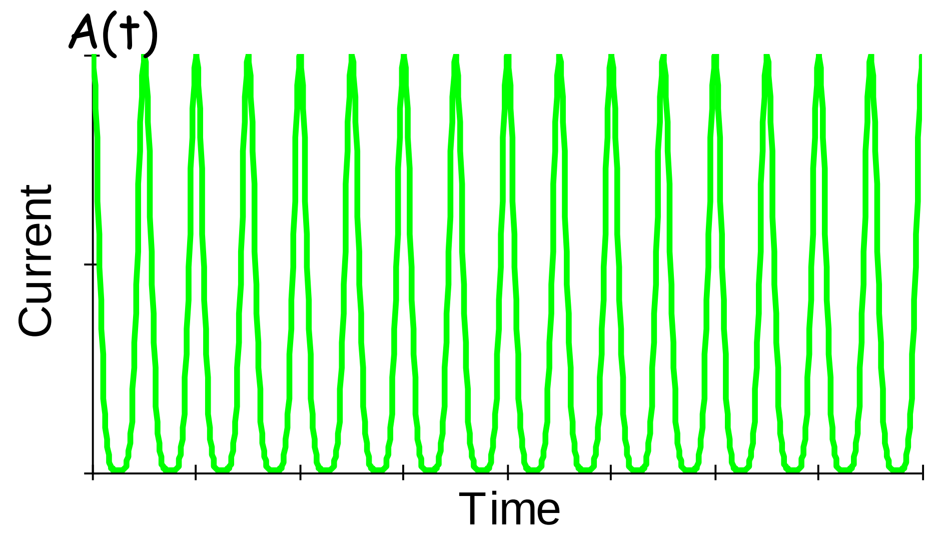

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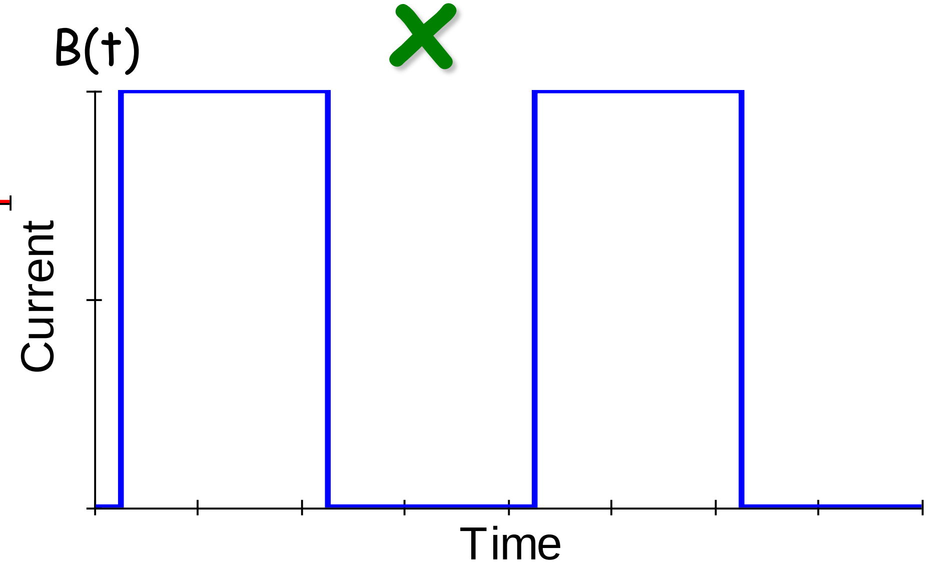


$i(t)=A(t)\times B(t)$

A Burst of Bunches in a Ring

$$A(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm(M\omega_{rev})t}$$

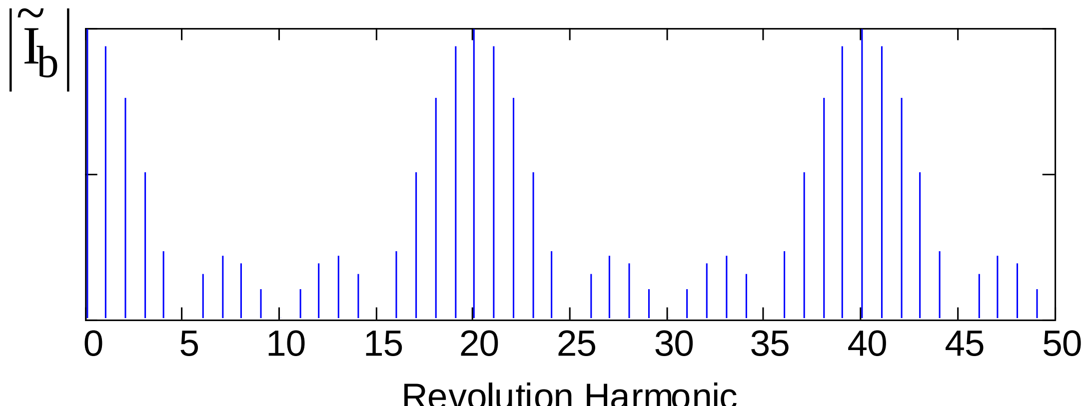
$$C_m = h\frac{qN_b}{T_{rev}} \int_{-T_{rev}/2}^{T_{rev}/2} f(\tau) e^{-jm(h\omega_{rev})\tau} d\tau$$

$$B(t) = \frac{M}{h} \sum_{n=-\infty}^{\infty} Sa\left(\frac{nM\pi}{h}\right) e^{jn\omega_{rev} t}$$

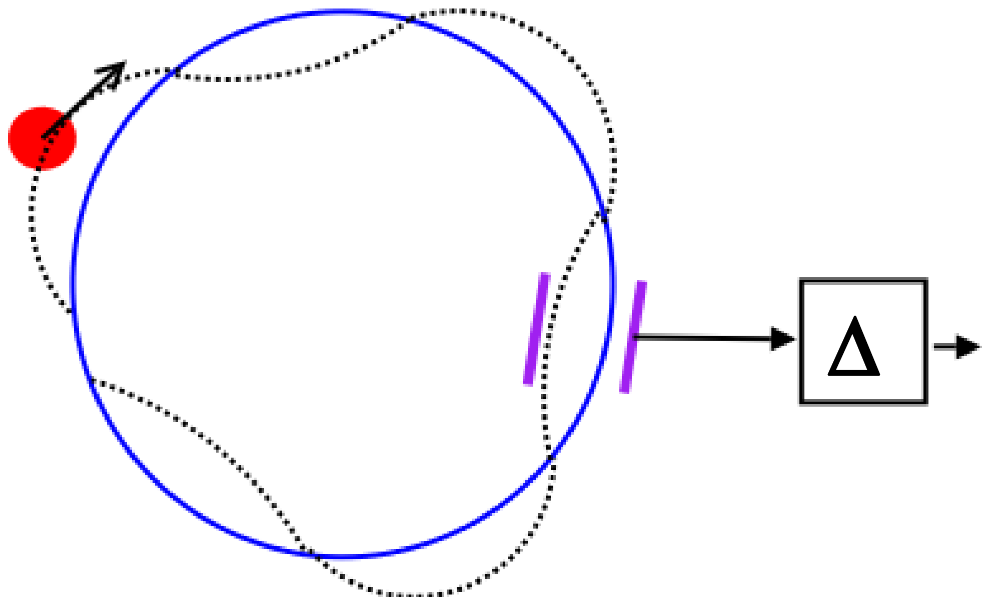
$$i_b(t) = \frac{M}{h} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m Sa\left(\frac{nM\pi}{h}\right) e^{j\omega_{rev}(mh+n)t}$$

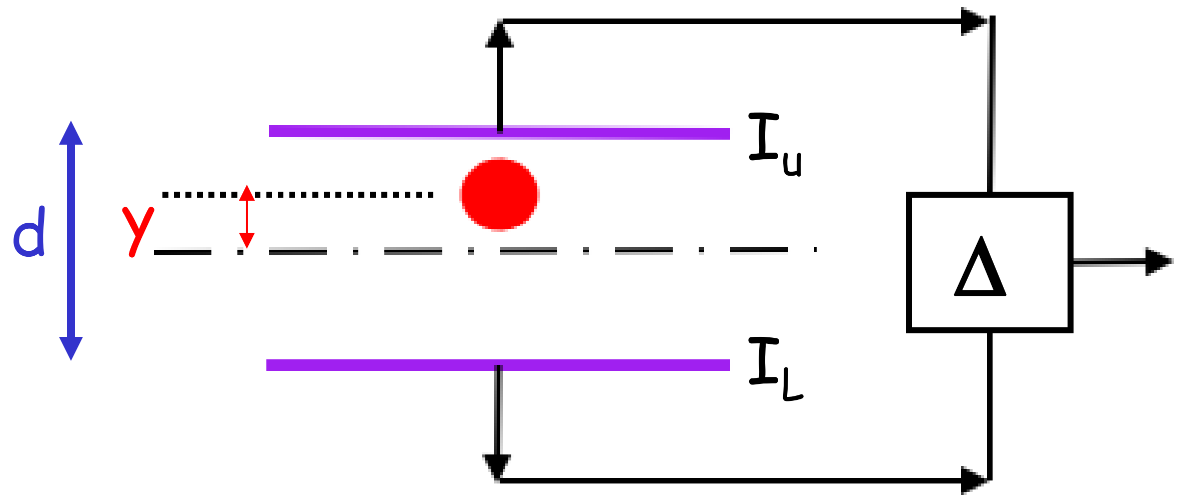
h=20

M=4



Betatron Motion





Betatron Motion

Image current collected on the pickup plates

$$I_U = \frac{I_b}{2}\left(1 + \frac{2y}{d}\right) \qquad I_L = \frac{I_b}{2}\left(1 - \frac{2y}{d}\right)$$

$$V_\Delta = \frac{Z_o}{\sqrt{2}}\left(I_U - I_L\right) = \sqrt{2} I_b Z_o \frac{y}{d}$$

Ideal power combiner

For a single particle in the ring:

$$i_p(t) = q \sum_{n=-\infty}^{\infty} \delta\left(t - nT_{rev}\right)$$

$$= \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\omega_{rev} t}$$

Betatron Motion

For a particle going through betatron oscillations

$$y = y_{co} + y_\beta \cos\left(Q\omega_{rev} t + \phi_\beta\right)$$

where Q is the tune
ϕ_β is the starting phase of the betatron oscillation
y_{co} is the closed orbit position

$$V_\Delta = \sqrt{2} Z_o \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} \left(\frac{y_{co}}{d} e^{jn\omega_{rev} t} + \frac{y_\beta}{d} e^{jn\omega_{rev} t} \cos\left(Q\omega_{rev} t + \phi_\beta\right) \right)$$

$$e^{jn\omega_{rev} t} \cos\left(Q\omega_{rev} t + \phi_\beta\right) = \frac{1}{2} e^{j\phi_\beta} e^{j(n+Q)\omega_{rev} t} + \frac{1}{2} e^{-j\phi_\beta} e^{j(n-Q)\omega_{rev} t}$$

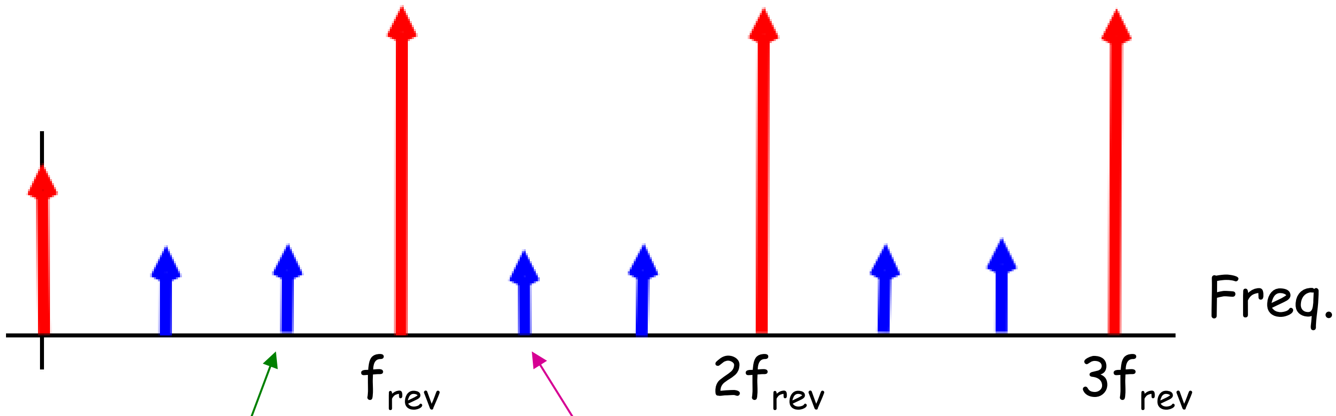
Betatron Oscillations

$$S(f) = 2\left(\frac{q}{T_{rev}}\right)^2 Z_o \left(\frac{y_{co}}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(f - nf_{rev})$$

$$+\frac{1}{2}\left(\frac{q}{T_{rev}}\right)^2 Z_o \left(\frac{y_\beta}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(f - (n - Q)f_{rev})$$

$$+\frac{1}{2}\left(\frac{q}{T_{rev}}\right)^2 Z_o \left(\frac{y_\beta}{d}\right)^2 \sum_{n=-\infty}^{\infty} \delta(f - (n + Q)f_{rev})$$

Betatron Oscillations



n-Q line for fract Q < 0.5
n-1+Q line for fract Q > 0.5

n+Q line for fract Q < 0.5
n+1-Q line for fract Q > 0.5

From one pickup, you cannot distinguish the integer part of the tune.
You also cannot tell if the tune is greater or less than 0.5

AM Modulation

Betatron oscillations has the same spectrum as Amplitude Modulation (AM)

$$v(t) = V_o\left(1 + m\cos(\omega_m t)\right)\cos(\omega_c t)$$

ω_m is the modulation frequency
ω_c is the carrier frequency
m is the modulation amplitude

Using the trig identity

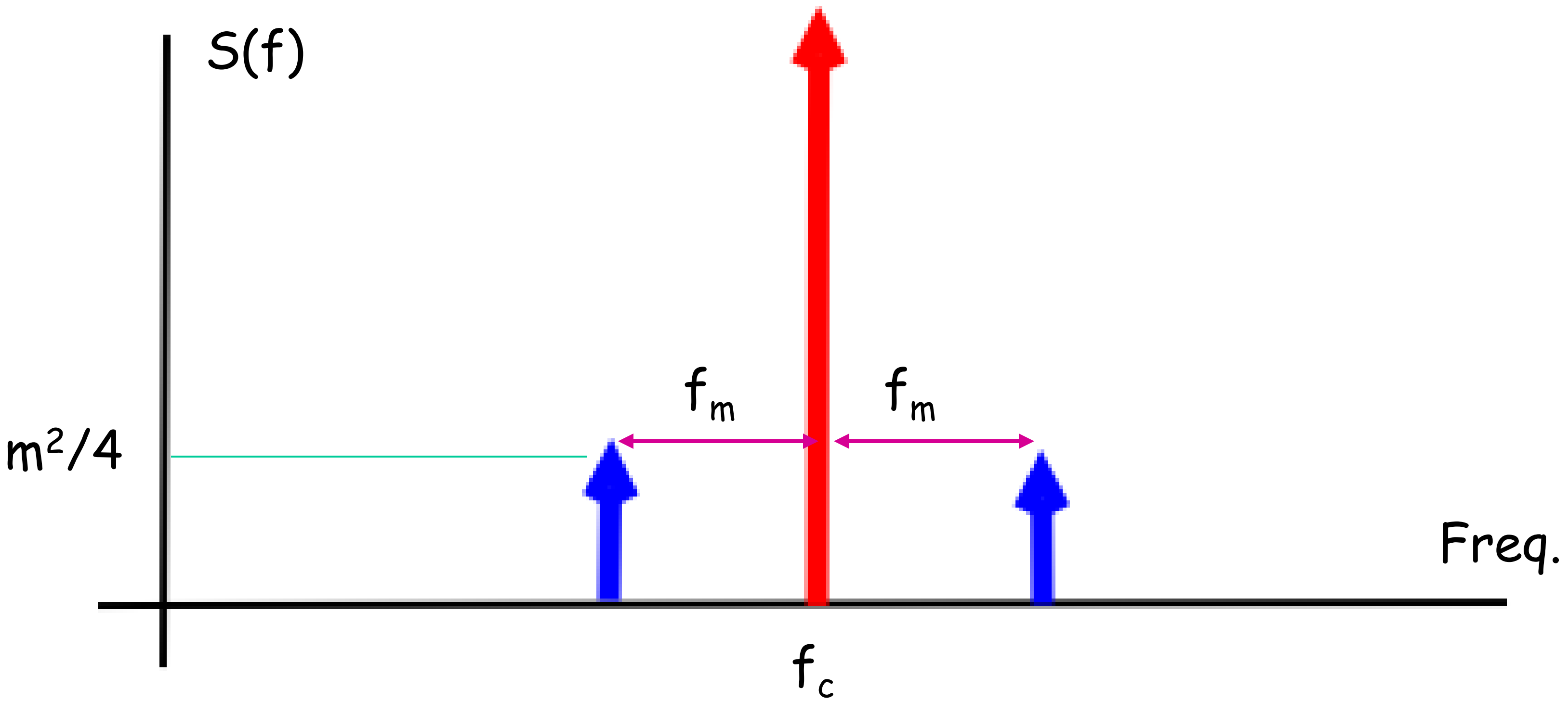
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$v(t) = V_o\cos(\omega_c t) + \frac{mV_o}{2}\cos((\omega_c + \omega_m)t) + \frac{mV_o}{2}\cos((\omega_c - \omega_m)t)$$

AM Modulation

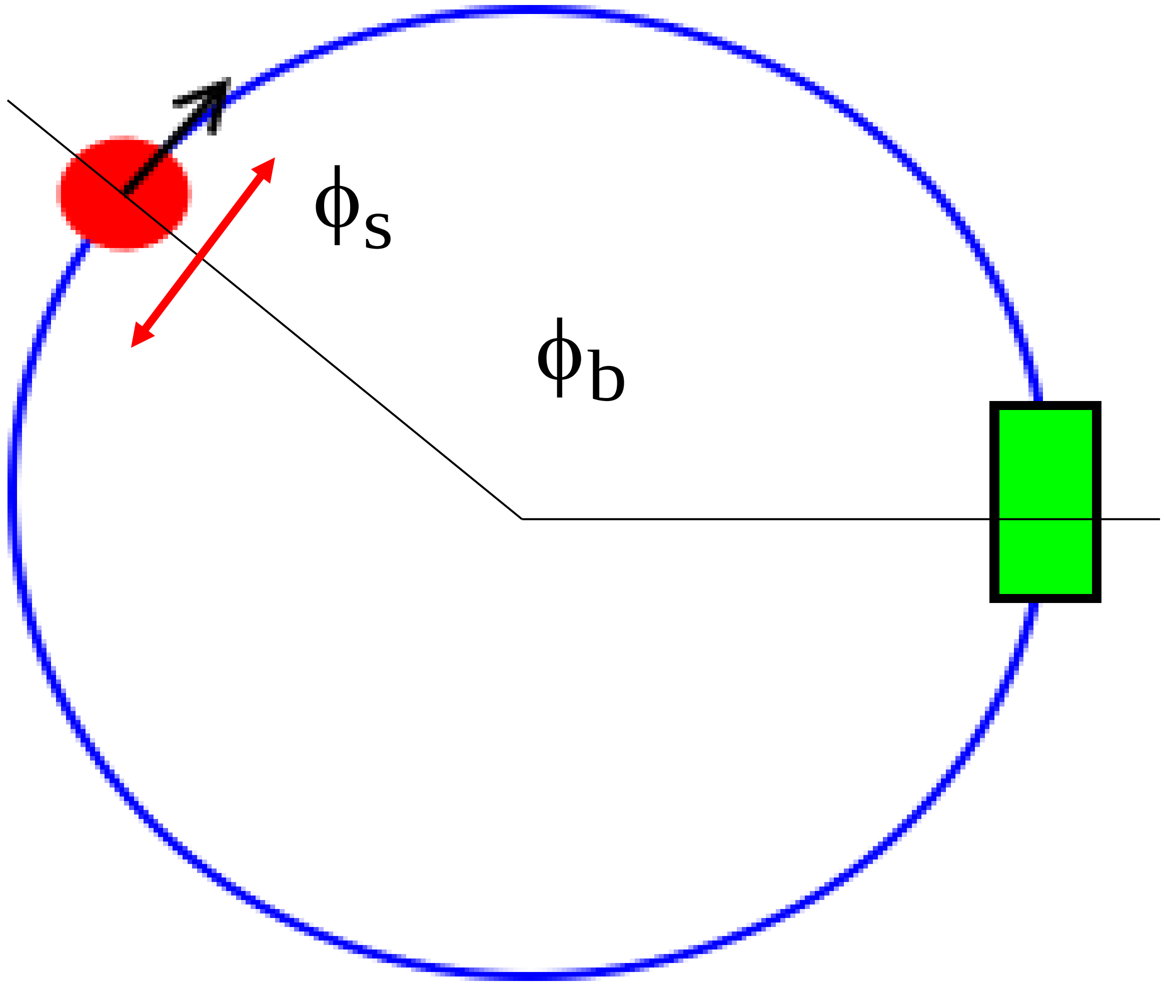
$$v(t) = V_o\left(1 + m\cos(\omega_m t)\right)\cos(\omega_c t)$$

$$v(t) = V_o\cos(\omega_c t) + \frac{mV_o}{2}\cos\left((\omega_c + \omega_m)t\right) + \frac{mV_o}{2}\cos\left((\omega_c - \omega_m)t\right)$$



Longitudinal Motion

A single particle undergoes synchrotron oscillations ϕ_s



Longitudinal Motion

The particle's longitudinal position can be described by an azimuthal phase around the ring. The particle traverses 2π radians in one trip around the ring.

$$i_p(\varphi) = q\omega_{rev} \sum_{n=-\infty}^{\infty} \delta(\varphi - 2n\pi)$$

This is a periodic function which can be expanded in a Fourier series:

$$i_p(\varphi) = \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\varphi}$$

Longitudinal Motion

But the longitudinal phase is a function of time which includes the time dependence of synchrotron oscillations

$$\varphi = \omega_{rev} t + \varphi_s \sin(\Omega_s t + \alpha_s)$$

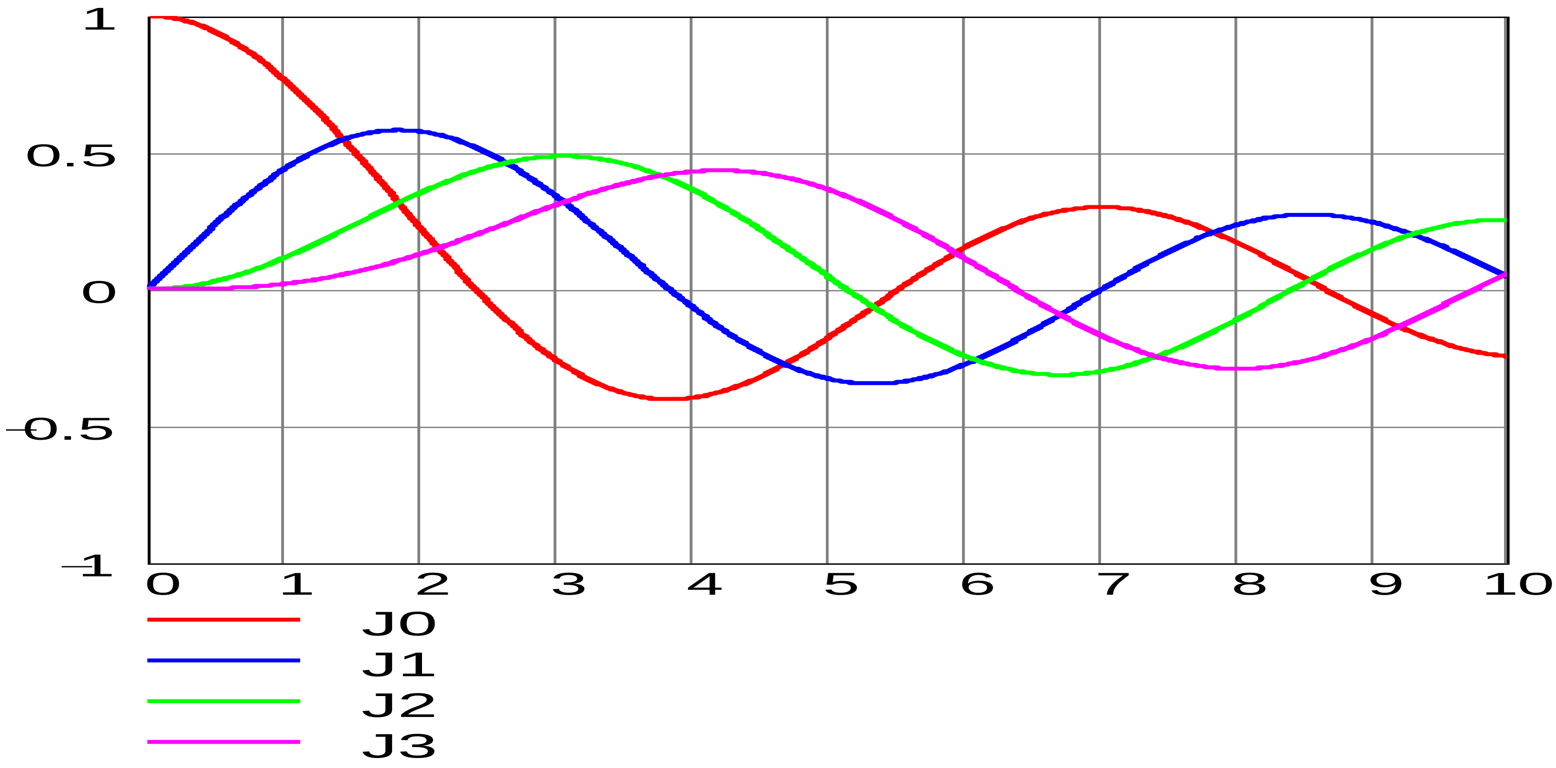
Where ϕ_s is the synchrotron oscillation amplitude,
Ω_s is the synchrotron frequency
α_s is the starting synchrotron phase

$$i_p(t) = \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} e^{jn\omega_{rev} t} e^{jn\varphi_s \sin(\Omega_s t + \alpha_s)}$$

Bessel Function Magic

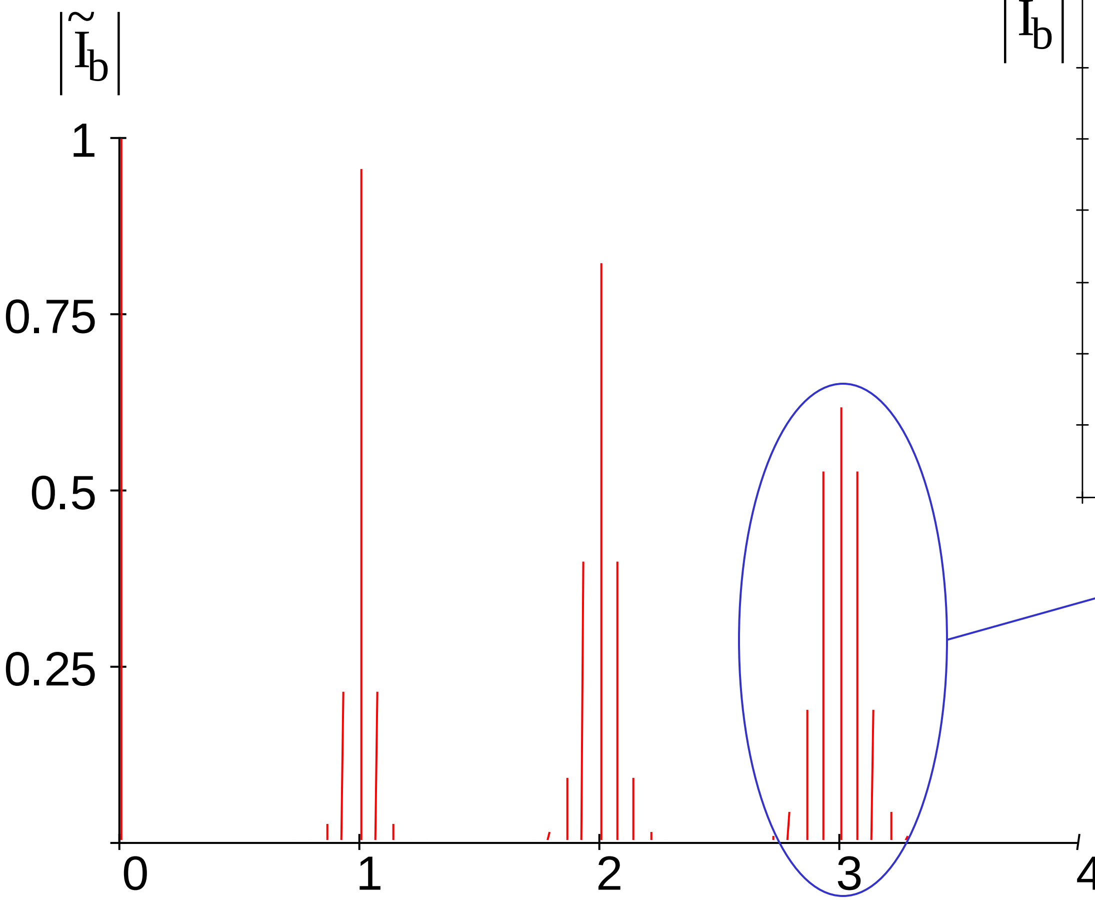
The complex exponential of a sine function can be "simplified" by using Bessel functions

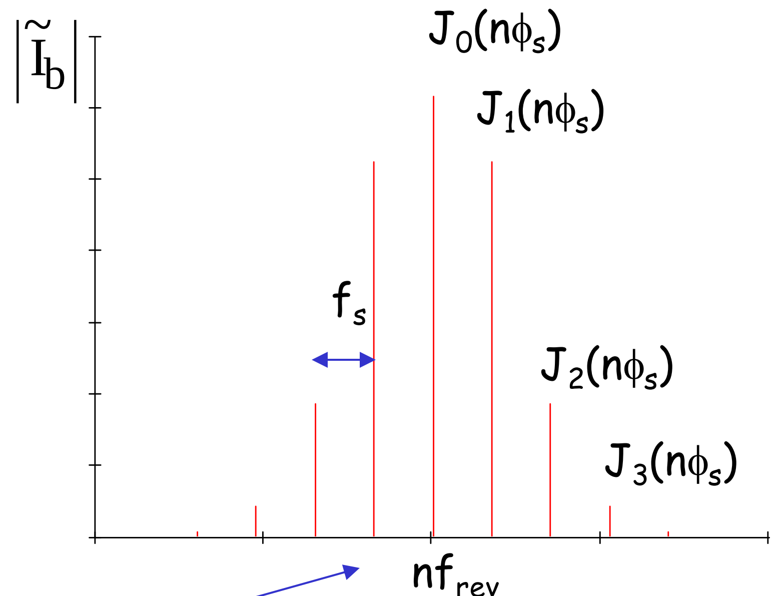
$$e^{jz\sin(x)} = \sum_{m=-\infty}^{\infty} J_m(z)e^{jmx}$$



Longitudinal Spectrum

$$i_p(t) = \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m(n\varphi_s) e^{jm\alpha_s} e^{j(n\omega_{rev} + m\Omega_s)t}$$





Frequency Modulation

Longitudinal oscillations has the same spectrum as frequency or phase modulation (FM or PM)

$$\omega = \omega_c + \omega_m \cos(\omega_s t)$$

ω_s is the modulation frequency
ω_c is the carrier frequency
ω_m is the modulation amplitude

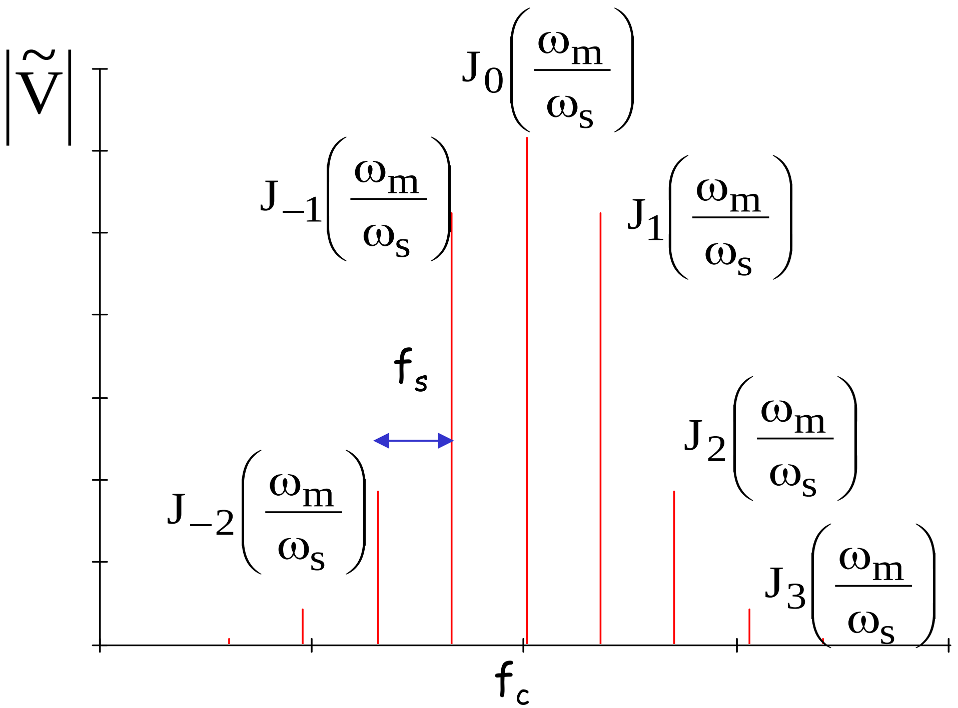
$$\frac{d\varphi}{dt} = \omega$$

$$\varphi = \omega_c t + \frac{\omega_m}{\omega_s} \sin(\omega_s t)$$

Frequency Modulation

$$v(t) = \cos\left(\omega_c t + \frac{\omega_m}{\omega_s}\sin(\omega_s t)\right)$$

$$v(t) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{\omega_m}{\omega_s}\right)\cos((\omega_c + m\omega_s)t)$$

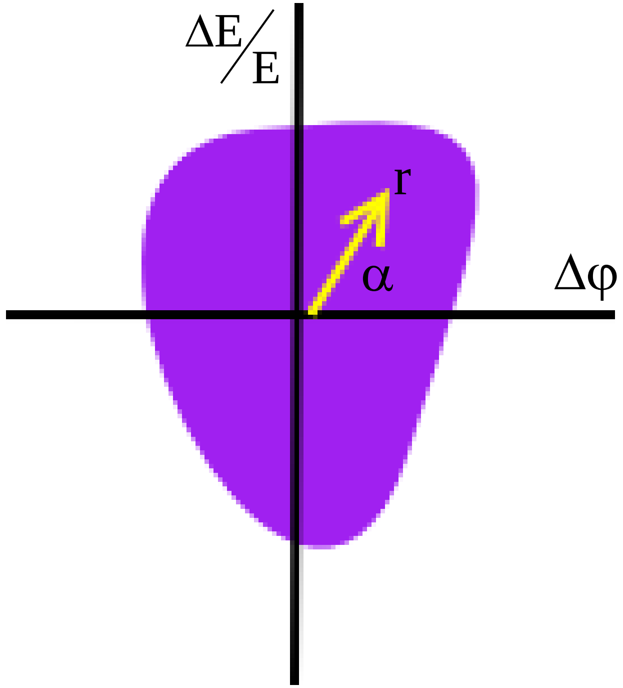


Multipole Distributions

The current for a single particle with synchrotron amplitude ϕ_s and phase α_s is:

$$i_p(\varphi_s, \alpha_s, t) = \frac{q}{T_{rev}} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m(n\varphi_s) e^{jm\alpha_s} e^{j(n\omega_{rev} + m\Omega_s)t}$$

What about a collection of particles in longitudinal phase space.



Each particle has a polar phase space coordinates radius r, angle α

EUROPEAN SPALLATION SOURCE

Multipole Distributions

The density in phase space must be periodic in α with a period of 2π

$$\psi(r,\alpha) = f(r)\sum_{k=-\infty}^{\infty} c_k e^{jk\alpha}$$

The number of particles in phase space is given by:

$$N_p = c_0 \int_0^{\infty} f(r) 2\pi r dr$$

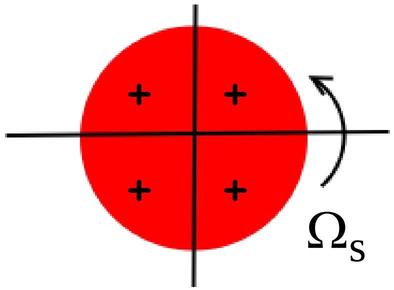
Also since $\psi(r,\alpha)$ must be real:

$$c_k = c_{-k}^{*}$$

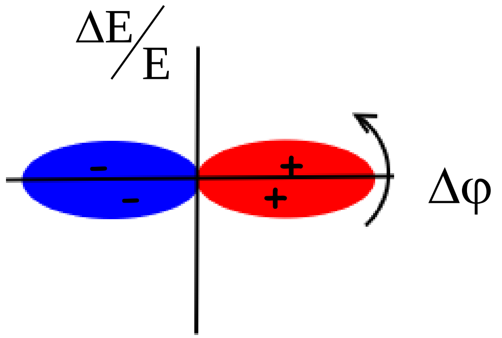
Multipole Distributions

The different values of k are multipoles. The multipoles spin around the origin at a the synchrotron frequency.

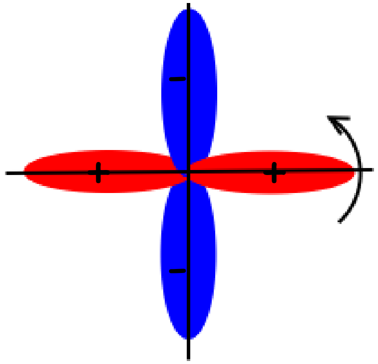
Monopole: $k = 0$



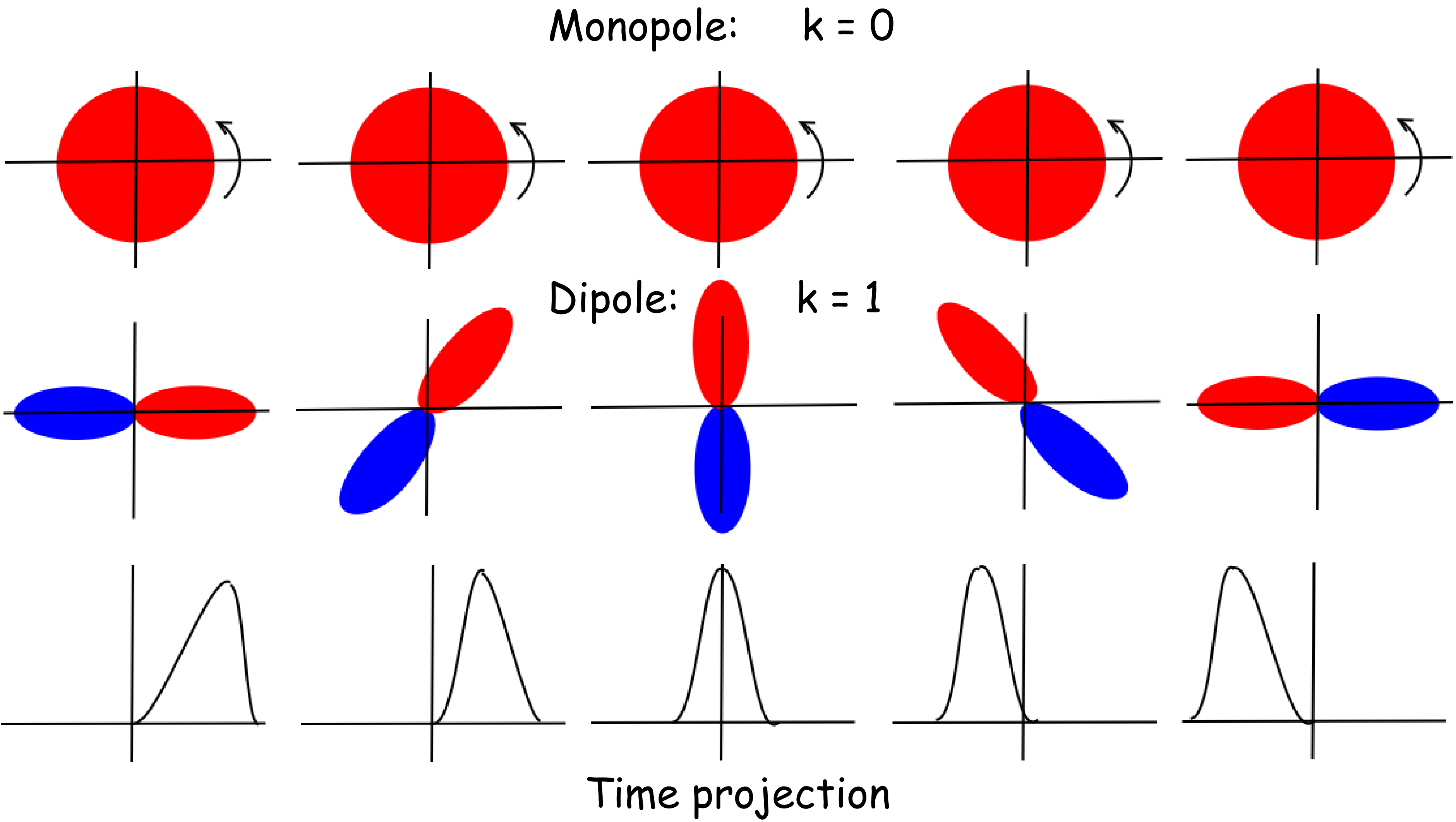
Dipole: $k = 1$



Quadrupole: $k = 2$

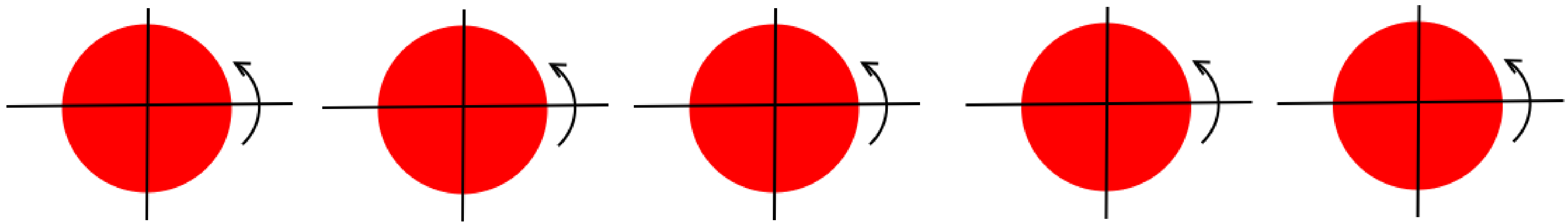


Multipole Distributions

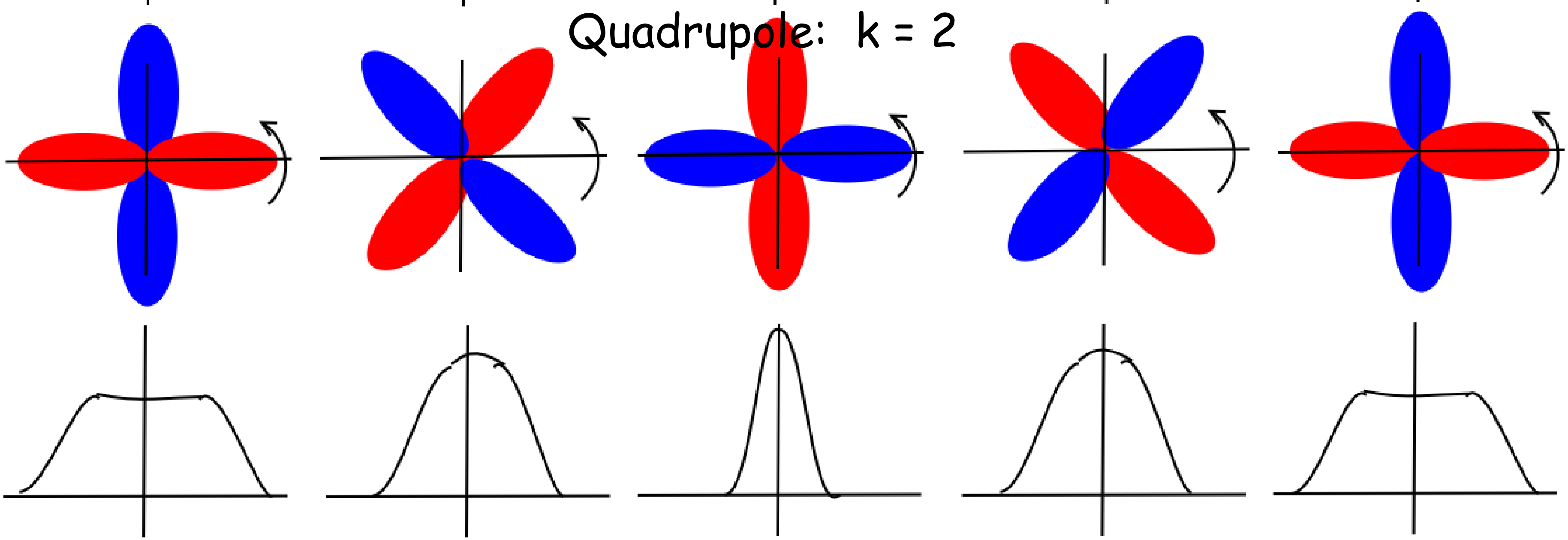


Multipole Distributions

Monopole: k = 0



Quadrupole: k = 2



Time projection

Multipole Distributions

The total current is found by integrating the contribution of each particle in phase space.

$$i_b(t) = \int_0^\infty r\,dr \int_0^{2\pi} \psi(r,\alpha)\, i_p(r,\alpha,t)\, d\alpha$$

$$i_b(t) = q\omega_{rev} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} F_k(n)\, c_k\, e^{j(n\omega_{rev} + k\Omega_s)t}$$

Where $F_k(n)$ is a frequency form factor

$$F_k(n) = \int_0^\infty J_k(nr)\, f(r)\, r\,dr$$

Multipole Distributions

Each synchrotron line in the spectrum corresponds to a different multipole mode oscillation