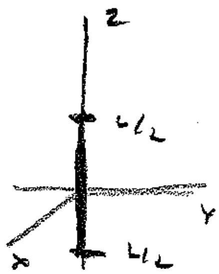
Consider a Uniform line source



$$\vec{J}(z') = I_0\, \delta(x-x')\, \delta(y-y')\, \hat{z} \quad \text{for } |z'| < \frac{L}{2}$$

$$\vec{J}(z') = 0 \quad |z'| > \frac{L}{2}$$

$$A_z = \frac{e^{-jkr}}{4\pi r} \int_{-L/2}^{L/2} I_0\, e^{jkz'\cos\theta}\, dz'$$

$$= \frac{e^{-jkr}}{4\pi r} I_0 \left[\frac{e^{jk(L/2)\cos\theta} - e^{-jk(\frac{L}{2})\cos\theta}}{jk\cos\theta} \right]$$

$$= \frac{I_0 L e^{-jkr}}{4\pi r} \; \frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\frac{kL}{2}\cos\theta}$$

$$\vec{E} = \hat{\Theta} \sqrt{\frac{\mu}{\varepsilon}}\, (jk\sin\theta)\, A_z$$

$$= \hat{\Theta}\, j\omega\mu \sin\theta\, A_z$$

$$= \underbrace{\hat{\Theta}}_{\substack{p \\ \text{polarization}}}\, j\omega\mu \frac{I_0 L}{4\pi r} e^{-jkr} \sin\theta\; \frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\frac{kL}{2}\cos\theta}$$

For a given r, define a normalized field pattern

$$\underbrace{F_p}_{\substack{p \\ \text{polarization}}}(\Theta, \phi) = \frac{E_p}{E_p(\text{max})}$$

For the uniform current antenna

$$F_\theta(\theta,\phi) = \sin\theta \; \frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\frac{kL}{2}\cos\theta}$$

$$F_p(\theta,\phi) = g_p(\theta,\phi)\, f_p(\theta,\phi)$$

$g_p(\theta,\phi)$ is called the <u>element factor</u> which is the pattern due to an infintesmal current element

$f_p(\theta,\phi)$ is called the pattern factor which is due to the integral over the current density

For the uniform current antenna

$$g_\theta(\theta,\phi) = \sin\theta$$

$$f_\theta(\theta,\phi) = \frac{\sin \frac{kL}{2}\cos\theta}{\frac{kL}{2}\cos\theta}$$

Radiation Intensity

$$\vec{S}(\theta,\phi) = \frac{1}{2}\mathrm{Re}\{E \times H^*\}$$

$$P_{rad}(r) = \int_0^{2\pi}\int_{-\pi}^{\pi} \vec{S}(\theta,\phi)\cdot\hat{r}\; r^2 \sin\theta \, d\theta \, d\phi$$

$$\vec{E}_{far} = E_\theta \hat{\theta} + E_\phi \hat{\phi}$$

$$\vec{H}_{far} = H_\theta \hat{\theta} + H_\phi \hat{\phi}$$

$$\vec{E}_{far} \times H^*_{far} = \left(E_\theta H_\phi^* - E_\phi H_\theta^*\right)\hat{r}$$

$$H_{\phi_{far}} = \frac{E_\theta}{\eta}$$

$$H_\theta = \frac{-E_\phi}{\eta}$$

$$P_r = \frac{1}{2\eta}\iint (|E_\theta|^2 + |E_\phi|^2\, r^2 \, d\Omega$$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Radiation Intensity is defined as

$$U(\theta,\phi) = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot r^2\hat{r}$$

$$\boxed{P_r = \iint_{\theta,\phi} U(\theta,\phi)\, d\Omega}$$

$U(\theta, \phi)$ has units of $\frac{\text{Watts}}{\text{steradian}}$

$$U(\theta, \phi) = U_m \, |F(\theta, \phi)|^2$$

$$U_m = U(\theta_{max}, \phi_{max})$$

$$P_r = \iint U(\theta, \phi) \, d\Omega$$

$$P_r = U_m \iint |F(\theta, \phi)|^2 \, d\Omega$$

Define $U_{ave} = \frac{P_r}{4\pi}$ ← Total angle of a sphere

$$U_{ave} = \frac{1}{4\pi} \iint U(\theta, \phi) \, d\Omega$$

<u>Ideal Dipole</u>

$$U(\theta, \phi) = \frac{1}{2} \left(\frac{I \Delta z}{4\pi} \right)^2 k \omega \mu \sin^2 \theta$$

$$U_m = \frac{1}{2} \left(\frac{I \Delta z}{4\pi} \right)^2 k \omega \mu$$

$$P_r = \left(\frac{k \omega \mu}{12\pi} \right) (I \Delta z)^2$$

Ideal Dipole

$$U_{ave} = \frac{P_r}{4\pi} = \frac{1}{3}\left(\frac{I\Delta z}{4\pi}\right)^2 k\omega\mu$$

$$U_{ave} = \frac{2}{3} U_m \quad \text{(ideal dipole)}$$

<u>Directive Gain</u>

$$\boxed{D(\theta, \phi) = \frac{U(\theta, \phi)}{U_{ave}}} \quad *$$

$$= \frac{U(\theta, \phi)}{\frac{1}{4\pi}\iint U(\theta, \phi)\, d\Omega}$$

$$= \frac{U_{max} |F(\theta, \phi)|^2}{\frac{1}{4\pi}\iint U_{max} |F(\theta, \phi)|^2 d\Omega}$$

$$\boxed{D(\theta, \phi) = \frac{|F(\theta, \phi)|^2}{\frac{1}{4\pi}\iint |F(\theta, \phi)|^2 d\Omega}}$$

Define <u>Antenna beam Solid angle</u> as

$$\boxed{\Omega_A = \iint |F(\theta, \phi)|^2 d\Omega}$$

Antenna Beam Solid Angle
is the effective angular coverage of the sky

For example if $|F(\theta,\phi)| = 1$ everywhere

$$\Omega_A = 4\pi \text{ steradians.}$$

$$D(\theta,\phi) = \frac{|F(\theta,\phi)|^2}{(\Omega_A/4\pi)}$$

Directivity is defined as

$$\boxed{D = \frac{U_m}{U_{ave}}}$$

since $|F(\theta_m, \phi_m)|^2 = 1$

$$D = \frac{4\pi}{\Omega_A}$$

Small Beam Angle $\Rightarrow$ Large Directivity

Large Beam Angle $\Rightarrow$ Small Directivity

For an Ideal Dipole

$$U_{ave} = \frac{2}{3} U_m$$

$$D = \frac{3}{2}$$

$$\Omega_A = \frac{2}{3}(4\pi)$$

Antenna Patterns

Plot $|F(\theta,\phi)|^2$ as a function of (θ,ϕ)

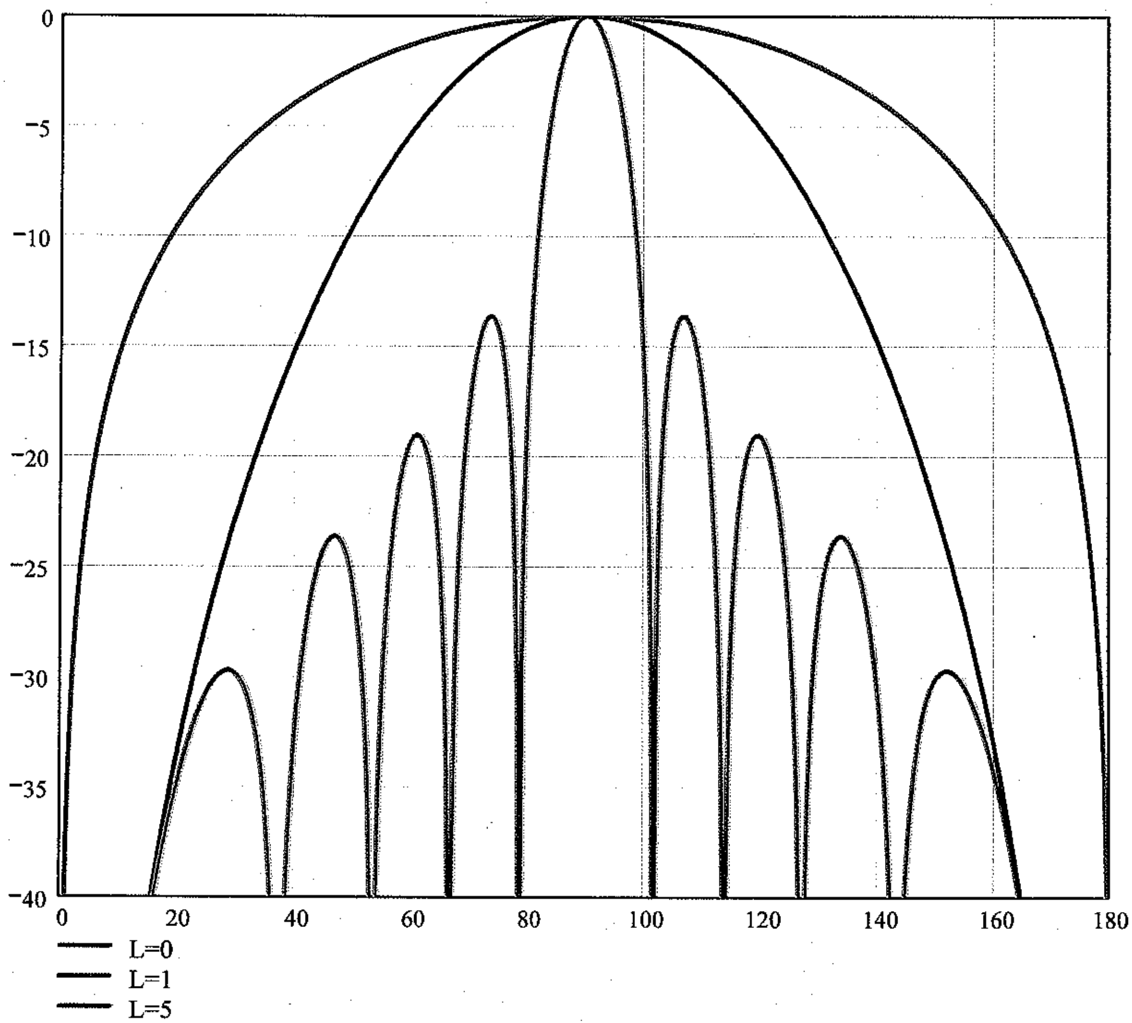
Ideal Dipole

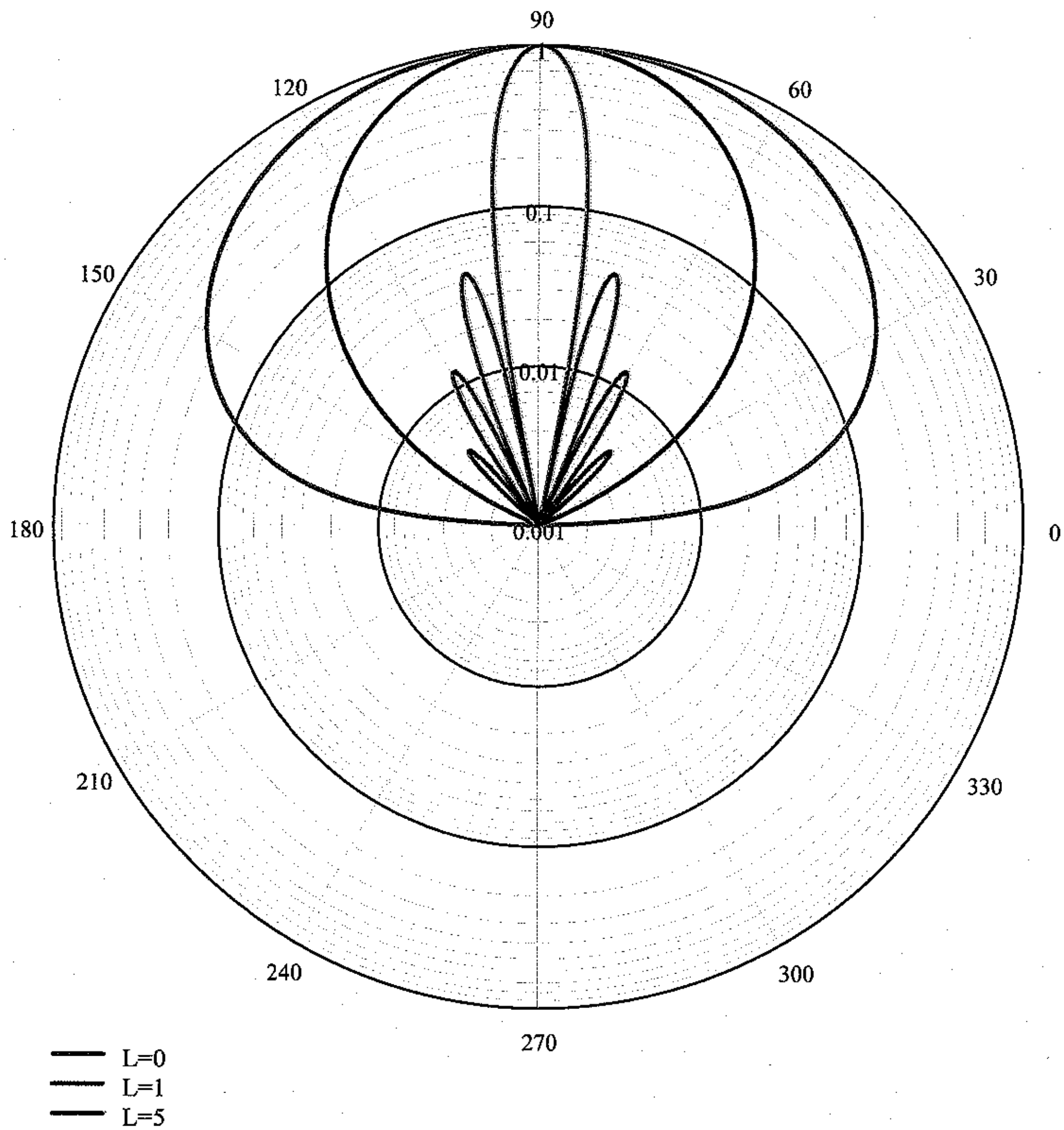
$$F(\theta,\phi) = \sin\theta$$

Uniform Dipole

$$F(\theta,\phi) = \sin\theta \frac{\sin\left(\frac{kL}{2}\cos\theta\right)}{\left(\frac{kL}{2}\cos\theta\right)}$$

$$= \frac{\sin\theta\left(\pi\frac{L}{\lambda}\cos\theta\right)}{\pi\frac{L}{\lambda}\cos\theta}$$


0
-5
-10
-15
-20
-25
-30
-35
-40
0
20
40
60
80
100
120
140
160
180
L=0
L=1
L=5


90
120
60
150
30
180
0
210
330
240
300
270
0.1
0.01
0.001
L=0
L=1
L=5

Reciprocity

$$\iiint_{v_a} \left(\vec{E}_b \cdot \vec{J}_a - \vec{H}_b \cdot \vec{M}_a\right) dv_a$$

$$= \iint \left(\vec{E}_a \cdot \vec{J}_b - \vec{H}_a \cdot \vec{M}_b\right) dv_b$$

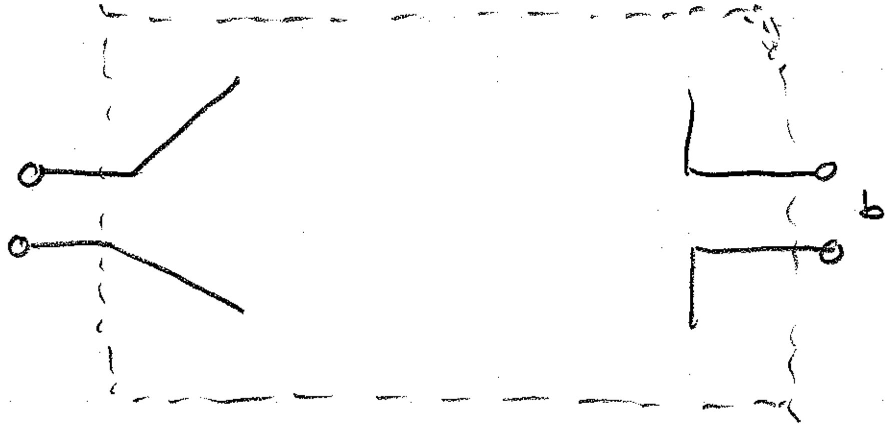
Consider current sources only

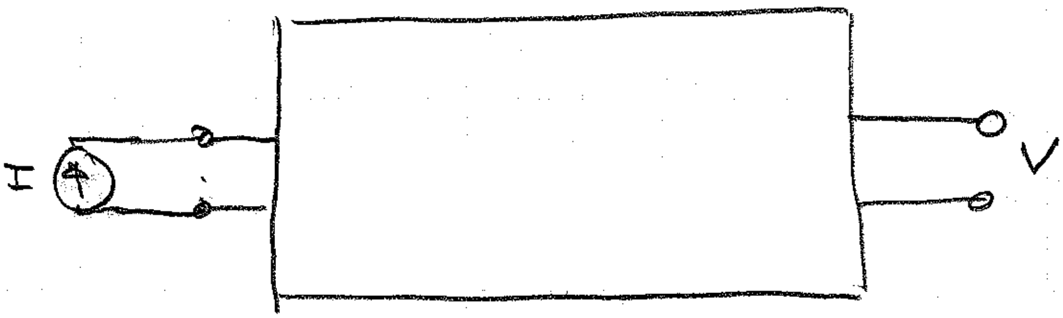
$$\iiint \underbrace{\vec{E}_b}_{\text{volts}/m} \cdot \underbrace{\vec{J}_a}_{A/m^2} \, dv_a = \iiint \vec{E}_a \cdot \vec{J}_b \, dv_b$$

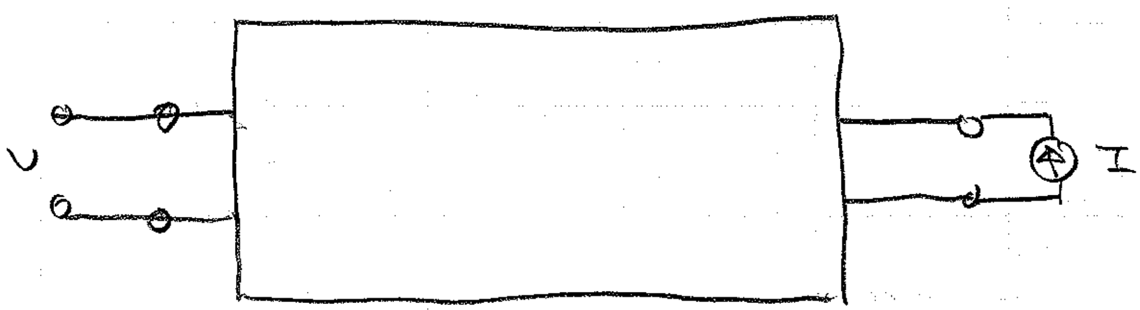
$$\vec{J}_A = I_A \, \delta(x - x_a') \, \delta(y - y_a') \, \hat{z}$$

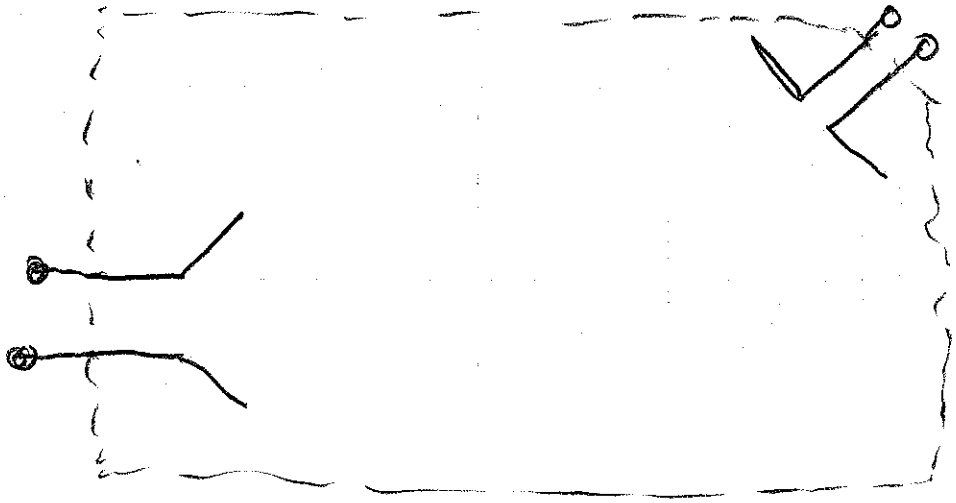
$$V_a\big|_b I_a = V_b\big|_a I_b$$

$V_a\big|_b$ = voltage at a due to sources at b



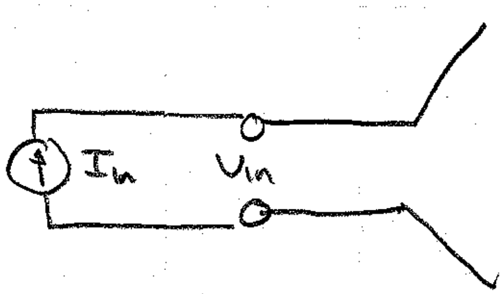

I
V


V
I



∴ Transmitting pattern is the same as the recieving pattern!

Antenna Impedance



$$V_{in} = (R_{in} + jX_{in})\, I_{in}$$

energy radiated
energy loss in wires

Energy stored in near field

$$P_{in} = \frac{1}{2} R_{in} |I_{in}|^2$$

$$P_{in} = P_r + P_{ohmic} = \frac{1}{2} R_{ri} |I_{in}|^2 + \frac{1}{2} R_{ohm} |I_{in}|^2$$

$$R_{ri} = \frac{2 P_r}{|I_{in}|^2}$$

$$P_r = \frac{1}{2} \iint_{S_{far}} (E_f \times H_f^*) \cdot d\vec{S}$$

For an ideal dipole

$$P_r = \frac{\omega \mu k}{12\pi} (I \Delta z)^2$$

$$= \frac{\omega \sqrt{\mu} \sqrt{\mu} \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}}}{12\pi} k (I \Delta z)^2$$

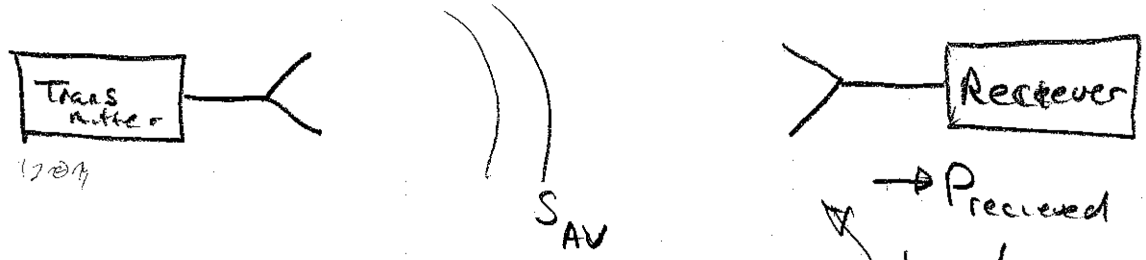
$$= \sqrt{\frac{\mu}{\varepsilon}} \frac{k^2}{12\pi} (I \Delta z)^2$$

$$= \eta \frac{1}{12\pi} \left(\frac{2\pi}{\lambda}\right)^2 (I \Delta z)^2$$

$$= \eta \frac{\pi}{3} \left(\frac{\Delta z}{\lambda}\right)^2 I^2$$

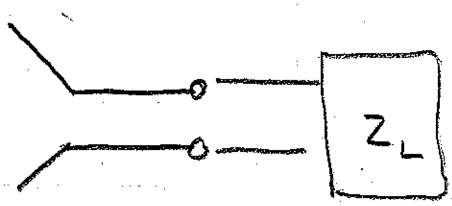
$$R_{ri} = \frac{2\pi}{3}\eta\left(\frac{\Delta z}{\lambda}\right)^2$$

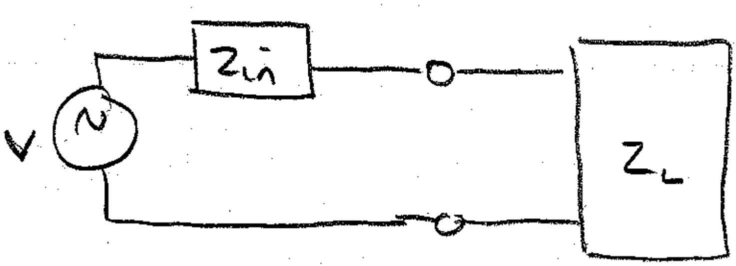
Maximum Effective Aperture



aligned to recieve max power

$$P_{recieved} = S_{AV}\, A_{e_{max}}$$





$$Z_{in} = R_{in} + j X_{in}$$

Then for max power transfer

$$Z_L = R_{in} - j X_{in}$$

Assume $R_{ohmic} = 0$

$$R_{in} = R_{ri}$$

$$I_{in} = \frac{V}{2R_{ri}} \quad \text{since } R_L \text{ set to } R_{ri}$$

$$P_{Rec} = \frac{1}{2}\frac{|V^2|}{4R_{ri}}$$

$$A_{em} S_{AV} = P_{rec}$$

Consider an Ideal dipole

$$V = E_{inc}\Delta Z$$

$$S_{av} = \frac{1}{2}\frac{E_{inc}^{2}}{\eta}$$

$$A_{em}\cancel{\frac{1}{2}}\frac{\cancel{E_{inc}^{2}}}{\eta} = \cancel{\frac{1}{2}}\frac{|\cancel{E_{inc}}\,\cancel{\Delta Z}|^2}{4\left(\frac{2\pi}{3}\eta\right)\left(\frac{\cancel{\Delta Z}}{\lambda}\right)^2}$$

$$A_{em} = \frac{3}{8\pi}\lambda^2$$

But $\Omega_{A_{Dipole}} = \frac{8\pi}{3}$

$$A_{em} = \frac{1}{\Omega_A} \lambda^2$$

$$\boxed{\Omega_A A_{Em} = \lambda^2}$$

Which holds true for any antenna!

Summary

$$\vec{E}_{far} = \vec{E}_0(\vec{k}) \frac{e^{-jkr}}{kr}$$

$$\vec{E}_0(\vec{k}) \cdot \hat{r} = 0$$

$$\vec{H}_{far} = \hat{r} \times \frac{1}{\eta} \vec{E}_{far}$$

$$r_{ff} > 2\left(\frac{L}{\lambda}\right) L$$

$L =$ size of structure

$$r_{ff} >> L$$

$$r_{ff} >> \lambda$$

$$P_r = \iint_{\theta\,\phi} U(\theta, \phi)\, d\Omega$$

$$U_{Avg} = \frac{1}{4\pi} \iint_{\theta\,\phi} U(\theta, \varphi)\, d\Omega$$

$$\Omega_A = \frac{\iint_{\theta\,\phi} U(\theta, \phi)\, d\Omega}{U_{max}}$$

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{U_{Avg}}$$

$$D = \frac{U_{max}}{U_{Avg}}$$

$$D = \frac{4\pi}{\Omega_A}$$

$$\Omega_A A_{em} = \lambda^2$$