TM_z Modes

If we set

$$A = \psi_m(x, y, z)\,\hat{z}$$

with

$$\nabla \times A = H$$

then $$H_z = 0$$

$$H_x = \frac{\partial \psi_m}{\partial y}$$

$$H_y = -\frac{\partial \psi_m}{\partial x}$$

$$H_z = 0$$

$$\frac{1}{j\omega\varepsilon} \nabla \times H = E$$

$$E_x = \frac{-k_z}{\omega\varepsilon} H_y = \frac{-k_z}{\omega\varepsilon} \frac{\partial \psi_m}{\partial x}$$

$$E_y = -\frac{k_z}{\omega\varepsilon} H_x = -\frac{k_z}{\omega\varepsilon} \frac{\partial \psi_m}{\partial y}$$

$$E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \frac{-1}{j\omega\varepsilon}\left(\frac{\partial^2 \psi_m}{\partial x^2} + \frac{\partial^2 \psi_m}{\partial y^2}\right)$$

$E_x = 0$ at $y = 0, b$

$E_y = 0$ at $x = 0, a$

$E_z = 0$ at $x = 0, a$ $\quad y = 0, b$

$$A = A_0 \hat{z} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

$$E_x = -A_0\left(\frac{k_z}{\omega\varepsilon}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

$$E_y = -A_0\left(\frac{k_z}{\omega\varepsilon}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

$$E_z = -j\left(\frac{\omega_c}{\omega}\right)\omega\varepsilon\mu A_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

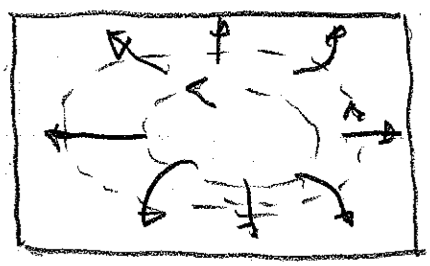
$$H_x = A_0 \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

$$H_y = -A_0 \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

$$H_z = 0$$

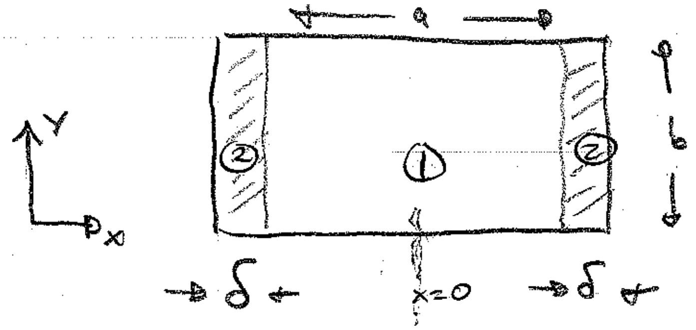
Note That you can't have a 1,0 or 0,1 TM mode

The simplest mode you can have is TM_{11}



Aternative Mode Sets

Consider the following problem



Where there is a dielectric slab on either side of the waveguide walls.

The tangential fields at the interface of the dielectric slab must be continuous. It would be impossible to satisfy these conditions with a TE^z or TM^z mode set.

However it is possible to satisfy the continuous boundary conditions if we solve the problem in TE^x or TM^x.

In the absdcence of the slab the TE_{10}^x mode will be the same the lowest order mode TE_{10}^z

$\therefore$ We will look at TE^x modes

$$\vec{F} = \hat{x} F_x(x,y) e^{-\gamma z}$$

where $\gamma = j k_z$

If there is loss in the dielectric γ will have a real part (or k_z will have an imaginary part).

$$\vec{E} = -\vec{\nabla} \times \vec{F}$$

$$\vec{H} = -\frac{1}{\hat{z}} \vec{\nabla} \times \vec{E}$$

$$\vec{E} = - \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ F_x & 0 & 0 \end{vmatrix}$$

$$E_x = 0$$

$$E_y = \gamma F_x$$

$$E_z = \frac{\partial F_x}{\partial y}$$

In region 1 $\quad E_z = 0$ at $y = 0, b$

$$F_{x_1} = F_1 \cos(f_1 x) \cos\left(\frac{n\pi y}{b}\right)$$

In region II

$$E_y = 0 \quad \text{at} \quad x = \frac{a}{2} + \delta$$

$$E_z = 0 \quad \text{at} \quad y = 0, b$$

$$F_{x_2} = F_2 \sin\left(f_2\left(x - \frac{a}{2} - \delta\right)\right) \cos\left(\frac{n\pi y}{b}\right)$$

Region I

$E_x = 0$

$$E_y = \gamma F_1 \cos(f_1 x) \cos\left(\frac{n\pi y}{b}\right)$$

$$E_z = -\left(\frac{n\pi}{b}\right) F_1 \cos(f_1 x) \sin\left(\frac{n\pi}{b}\right)$$

Region II

$E_x = 0$

$$E_y = \gamma F_2 \sin\left(f_2\left(x - \frac{a}{2} - \delta\right)\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$E_z = -\left(\frac{n\pi}{b}\right) F_2 \sin\left(f_2\left(x - \frac{a}{2}\ \delta\right)\right) \sin\left(\frac{n\pi}{b}\right)$$

Lets consider the TE^x_{10} mode first

Region I

$E_x = 0$

$E_y = \gamma F_1 \cos(f_1 x)$

$E_z = 0$

Region II

$E_x = 0$

$E_y = \gamma F_2 \sin\left(f_2\left(x - \frac{a}{2} - \delta\right)\right)$

$E_z = 0$

$$H = -\frac{1}{Z} \vec{\nabla} \times \vec{E}$$

$$= -\frac{1}{Z} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ 0 & E_y & 0 \end{vmatrix}$$

$H_x = -\frac{1}{Z} \gamma E_y$

$H_y = 0$

$H_z = -\frac{1}{Z} \frac{\partial E_y}{\partial x}$

Region I

$$H_x = -\frac{\gamma^2}{\hat{Z}_1} F_1 \cos(f_1 x)$$

$$H_y = 0$$

$$H_z = \frac{\gamma f_1}{\hat{Z}_1} F_1 \sin(f_1 x)$$

Region II

$$H_x = -\frac{\gamma^2}{\hat{Z}_2} F_2 \sin\left(f_2\left(x - \frac{a}{2} - \delta\right)\right)$$

$$H_y = 0$$

$$H_z = -\frac{\gamma f_2}{\hat{Z}_2} F_2 \cos f_2\left(x - \frac{a}{2} - \delta\right)$$

at $x = \frac{a}{2}$ $\quad E_{y_1} = E_{y_2}$

$$\gamma F_1 \cos\left(f_1 \frac{a}{2}\right) = -\gamma F_2 \sin(f_2 \delta)$$

at $x = \frac{a}{2}$ $\quad H_{z_1} = H_{z_2}$

$$\frac{\gamma f_1}{\hat{Z}_1} F_1 \sin\left(f_1 \frac{a}{2}\right) = -\frac{\gamma f_2}{\hat{Z}_2} F_2 \cos(f_2 \delta)$$

$$\frac{f_1}{Z_1} \tan\left(f_1 \frac{a}{2}\right) = \frac{f_2}{Z_2} \frac{1}{\tan(f_2 \delta)}$$

$$\frac{f_1}{f_2} \frac{Z_2}{Z_1} \tan\left(f_1 \frac{a}{2}\right) \tan f_2 \delta = 1$$

$$\gamma^2 - f_1^2 - \hat{y}_1 \hat{z}_1 = 0$$

$$\gamma^2 - f_2^2 - \hat{y}_2 \hat{z}_2 = 0$$

$$f_1^2 + \hat{y}_1 \hat{z}_1 = f_2^2 + \hat{y}_2 \hat{z}_2$$

$$\boxed{f_2^2 = f_1^2 + (y_1 z_1 - y_2 z_2)}$$

Some Assumptions

$$\delta \ll 1$$

$$f_1 \frac{a}{2} = \frac{\pi}{2} + \Delta$$

$$f_1 = \frac{\pi}{a} + \frac{2\Delta}{a}$$

$$(\pi + 2\Delta) \frac{\delta}{a} \frac{\hat{z}_2}{z_1} \left(\frac{-1}{\Delta}\right) = 1$$

$$\left(\frac{\pi}{\Delta} + 2\right) \frac{\delta}{a} \frac{z_2}{z_1} = -1$$

$$\left(\frac{\pi}{\Delta} + 2\right) = -\frac{a}{\delta} \frac{z_1}{z_2}$$

$$-\frac{\pi}{\Delta} = -2 + \frac{a}{\delta}\frac{Z_1}{Z_2}$$

$$\Delta = \frac{-\pi}{2 + \frac{a}{\delta}\frac{Z_1}{Z_2}}$$

$$\Delta = -\pi\frac{\delta}{a}\frac{Z_2}{Z_1}$$

$$\gamma^2 = f_1^2 - \omega^2\mu_1\varepsilon_1$$

$$f_1^2 = \left(\frac{\pi}{a} + \frac{2\Delta}{a}\right)^2$$

$$= \left(\frac{\pi}{a}\right)^2\left(1 + \frac{2\Delta}{\pi}\right)^2$$

$$= \left(\frac{\pi}{a}\right)^2\left(1 - 2\frac{\delta}{a}\frac{Z_2}{Z_1}\right)^2$$

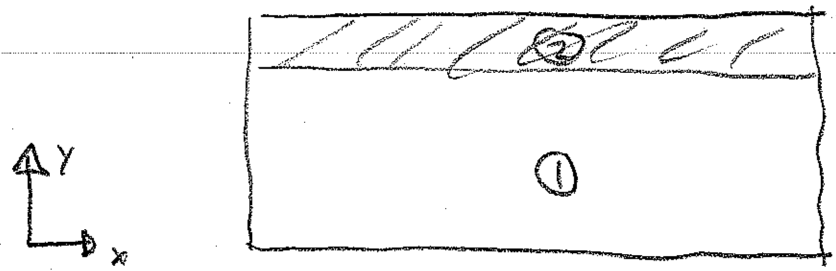
$$f_1^2 \approx \left(\frac{\pi}{a}\right)^2\left(1 - 4\frac{\delta}{a}\frac{Z_2}{Z_1}\right)$$

$$\gamma^2 = \left(\frac{\pi}{a}\right)^2\left(1 - 4\frac{\delta}{a}\frac{Z_2}{Z_1}\right) - \omega^2\mu\varepsilon$$

$$\gamma^2 = \left(\frac{\pi}{a}\right)^2\left(1 - 4\frac{\delta}{a}\frac{\mu_2}{\mu_1}\right) - \omega^2\mu\varepsilon$$

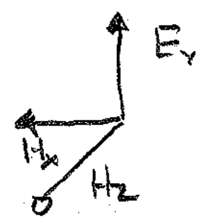
Note that the perturbation only depends on μ $\Rightarrow$ Why?

How Would one solve this geometry?



We would use TE^y or TM^y.

For the unloaded case, the TE_{10}^z mode (fundamental) has



$\therefore$ TM_{10}^y mode would reduce to TE_{10}^z mode in the limit of no loading

Cylindrical Waveguides.

TE^z mode

$$\nabla^2 F_z + k^2 F_z = 0$$

$$F_z = F_{z_+}(\rho, \varphi)\, e^{-jk_z z}$$

$$\frac{\partial^2 F_{z_+}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_{z_+}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F_{z_+}}{\partial \varphi^2} - k_z^2 F_{z_+} + k^2 F_{z_+} = 0$$

$$F_{z_+} = F_{z_\rho}(\rho)\left[A \sin n\varphi + B \cos n\varphi\right]$$

because F_{z_+} must be periodic in φ in intervals of 2π

$$\frac{\partial^2 F_{z_\rho}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_{z_\rho}}{\partial \rho} - \frac{n^2}{\rho^2} F_{z_\rho} = \left(k_z^2 - k^2\right) F_{z_\rho}$$

$$k_c^2 + k_z^2 = k^2$$

$$\frac{\partial^2 F_{z_\rho}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_{z_\rho}}{\partial \rho} + \left(k_c^2 - \frac{n^3}{\rho^2}\right) F_{z_\rho} = 0$$

$$u = k_c \rho$$

$$du = k_c\, d\rho$$

$$k_c^2 \frac{\partial^2 F_{z\rho}}{du^2} + \frac{k_c^2}{u}\frac{\partial^2 F_{z\rho}}{du} + \left(k_c^2 - k_c^2 \frac{n^2}{u^2}\right) F_{z\rho} = 0$$

$$\frac{\partial^2 F_{z\rho}}{\partial u^2} + \frac{1}{u}\frac{\partial F_{z\rho}}{\partial u} + \left(1 - \frac{n^2}{u^2}\right) F_{z\rho} = 0$$

This is Bessel's eqn.

$$F_{z\rho} = C\, J_n(k_c\rho) + \cancel{D\, N_n(k_c\rho)}$$

unbounded at $\rho = 0$

$$F_{z+} = F_{z_0} e^{jn\phi} J_n(k_c\rho)$$

$$\vec{E} = -\nabla \times \vec{F}$$

$$E_\rho = -\frac{1}{\rho}\frac{\partial F_z}{\partial \phi}$$

$$E_\phi = \frac{\partial F_z}{\partial \rho} = k_c \frac{\partial F_z}{\partial (k_c\rho)}$$

$$E_z = 0$$

$$E_\rho = \frac{-jn}{\rho} F_{zo} e^{jn\phi} J_n(k_c \rho)$$

$$E_\phi = k_c F_{zo} e^{jn\phi} J_n'(k_c \rho)$$

$$J_n'(k_c a) = 0$$

$$\chi_{n\ell}' = \text{zero's of } J_n'(u)$$

i.e $$J_n'(\chi_{n\ell}') = 0$$

$$k_c = \frac{\chi_{n\ell}'}{a}$$

$$F_{z_+} = F_{zo} e^{jn\phi} J_n\left(\frac{\chi_{n\ell}'}{a} \rho\right)$$

Zero's of J_n'

$\ell =$	$n =$ 0	1	2	3
1	3.83	1.84	3.05	4.20
2	7.02	5.33	6.71	8.02
3	10.17	8.54	9.97	11.35

TE mode

$$E_\rho = -\frac{n}{\rho} F_{z0} \sin(n\varphi)\, J_n(k_c \rho)$$

$$E_\varphi = k_c F_{z0} \cos(n\varphi)\, J_n'(k_c \rho)$$

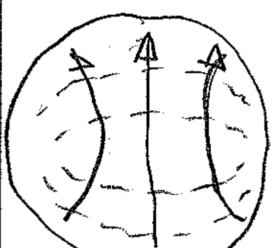
$$E_z = 0$$

$$H_\rho = -j\frac{k_z k_c}{\hat{Z}} F_{z0} \cos(n\varphi)\, J_n'(k_c \rho)$$

$$H_\varphi = j\frac{k_z}{\hat{Z}}\, n\, F_{z0} \sin(n\varphi) \frac{1}{\rho} J_n(k_c \rho)$$

$$H_z = \frac{k_c^2}{\hat{Z}} F_{z0} \cos(n\varphi)\, J_n(k_c \rho)$$

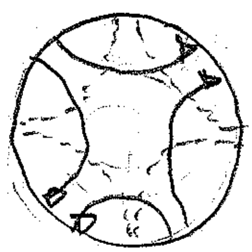
$k_c a = 1.841$



TE_{11}

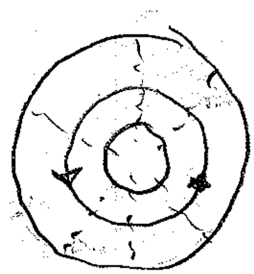
lowest mode

$k_c a = 3.054$



TE_{21}

$k_c a = 3.82$



TE_{01}

Magic Mode

Cylindrical Waveguide TM^z mode

$$A = A_{z_T}(x,y)e^{-jk_z z}\,\hat{z}$$

$-k_z^2$

$$H_\rho = \frac{1}{\rho}\frac{\partial A_{z_T}}{\partial \varphi} \qquad E_\rho = -j\frac{k_z}{\hat{y}}\frac{\partial A_{z_T}}{\partial \rho}$$

$$H_\varphi = -\frac{\partial A_{z_T}}{\partial \rho} \qquad E_\varphi = -j\frac{k_z}{\hat{y}}\frac{1}{\rho}\frac{\partial A_{z_T}}{\partial \varphi}$$

$$H_z = 0 \qquad E_z = \frac{k_c^2}{\hat{y}}A_z$$

$$E_z = \frac{k_c^2}{\hat{y}}F_0\cos(n\varphi)\,J_n(k_c\rho)$$

E_z must be zero at $\rho = a$

$$k_c a = \chi_{n\ell}$$

where $J_n(\chi_{n,\ell}) = 0$

ℓ \ n	0	1	2	3
1	2.40	3.83	5.14	6.38
2	5.52	7.02	8.42	9.76
3	8.65	10.17	11.62	13.02

TM mode

$$E_\rho = -j \frac{k_z k_c}{\hat{y}} A_{z0} \cos(n\phi) J_n'(k_c \rho)$$

$$E_\phi = j \frac{k_z}{\hat{y}} n A_{z0} \sin(n\phi) \frac{J_n(k_c \rho)}{\rho}$$

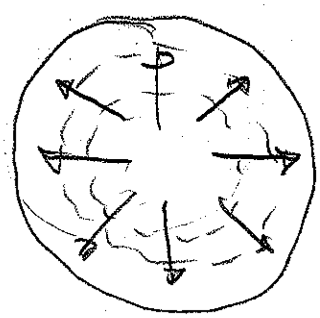
$$E_z = \frac{k_c^2}{\hat{y}} A_{z0} \cos(n\phi) J_n(k_c \rho)$$

$$H_\rho = \frac{-n}{\rho} A_{z0} \sin(n\phi) J_n(k_c \rho)$$

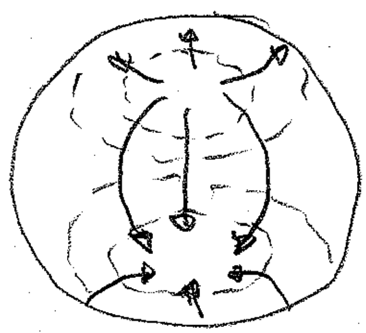
$$H_\phi = -k_c A_{z0} \cos(n\phi) J_n'(k_c \rho)$$

$$H_z = 0$$

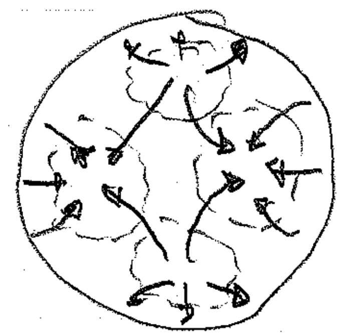
$k_c a = 2.405$ $k_c a = 3.83$ $k_c a = 5.13$



TM_{01}



TM_{11}



TM_{21}

Beam pipe mode

Sources in a Waveguide

The fields in a waveguide can be expanded as a sum over all the waveguide modes

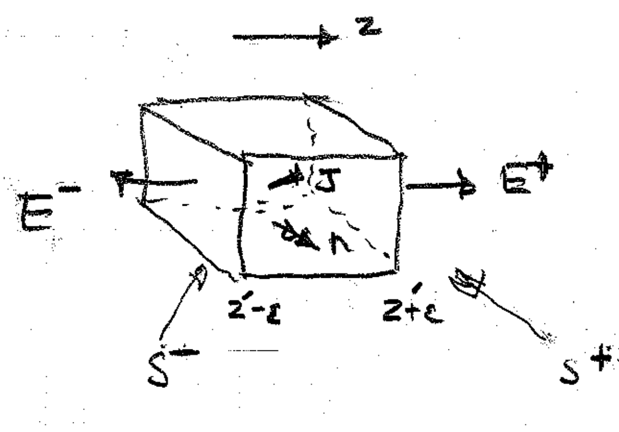
$$\vec{E}^+ = \sum_n C_n^+(\hat{e}_{t_n} + \hat{e}_{z_n})e^{-jk_{z_n}z}$$

$$\vec{E}^- = \sum_n C_n^-(\hat{e}_{t_n} - \hat{e}_{z_n})e^{jk_{z_n}z}$$

$$\vec{H}^+ = \sum_n C_n^+(\hat{h}_{t_n} + \hat{h}_{z_n})e^{-jk_{z_n}z}$$

$$\vec{H}^- = \sum_n C_n^-(-\hat{h}_{t_n} + \hat{h}_{z_n})e^{jk_{z_n}z}$$

Consider a volume of sources



Reciprocity States

$$\oint\!\!\oint_S (\vec{E}^a \times \vec{H}^b) - (E^b \times H^a) \cdot \hat{n}\, dS$$

$$= \iiint_V (\vec{E}^a \cdot \vec{J}^b - \vec{H}^a \cdot \vec{M}^b - \vec{E}^b \cdot \vec{J}^a + \vec{H}^b \cdot \vec{M}^a)\, dv$$

Let the (a) field be due to sources

$$\vec{E}_a = E \qquad \vec{H}^a = \vec{H} \qquad \vec{J}^a = \vec{J} \qquad \vec{M}^a = \vec{M}$$

Let the (b) field be due to <u>one</u> of the reverse modes

$$E^b = (\hat{e}_{tm} - \hat{e}_{zm}) e^{jk_{zm} z}$$

$$H^b = (-\hat{h}_{tm} + h_{zm}) e^{jk_{zm} z}$$

$$\vec{J}^b = 0$$

$$\vec{M}^b = 0$$

Source Integral becomes

$$-\iiint_V \left[(\hat{e}_{tm} - \hat{e}_{zm}) \cdot \vec{J} + (\hat{h}_{tm} - \hat{h}_{zm}) \cdot M \right] e^{jk_{zm} z}\, dV$$

Since the tangential electric fields on the wall's are zero, the surface integral be omes

$$\iint_{S^-} \left(\vec{E}^- \times \vec{h}_m^-\right) - \left(\hat{e}_m^- \times \vec{H}^-\right) \cdot (-\hat{z})\, dS$$

$$+ \iint_{S^+} \left(\vec{E}^+ \times \vec{h}_m^-\right) - \left(\hat{e}_m^- \times \vec{H}^+\right) \cdot \hat{z}\, dS$$

$$\left(E^- \times h_m^-\right) \cdot \hat{z} = \sum_n C_n^- \left(\hat{e}_{+n} + \hat{e}_{z_n}\right) X \left(-h_{+m} + h_{z_m}\right) e^{jk_{z_m} z} e^{jk_{z_n} z} \cdot \hat{z}$$

$$= -\sum_n C_n^- \left(\hat{e}_{+n} \times \hat{h}_{+m}\right) \cdot \hat{z}\ e^{jk_{z_m} z} e^{jk_{z_n} z}$$

$$\iint_{S^-} \left(E^- \times h_m^-\right) \cdot \hat{z} = -C_m \iint_{S^-} \left(e_{+m} \times h_{+m}\right) \cdot \hat{z}\, e^{j2k_{z_m} z}\, dS$$

because

$$\iint \left(\hat{e}_{+n} \times \hat{h}_{+m}\right) \cdot \hat{z}\, ds = 0$$

for $m \neq n$

Power orthogonal

$$(\hat{e}_m^- \times H^-)\cdot\hat{z} = \hat{z}\cdot\sum_n C_n^- (\hat{e}_{tm} - \hat{e}_{zm}) \times (-\hat{h}_{tn} + \hat{h}_{zn}) e^{jk_{zm}z} e^{jk_{zn}z}$$

$$= -\sum_n C_n^- (\hat{e}_{tm} \times \hat{h}_{tn})\cdot\hat{z}\; e^{jk_{zm}z}\; e^{jk_{zn}z}$$

$$\iint_{S_-} (\hat{e}_m^- \times \vec{H}^-)\cdot\hat{z}\, dS = -C_m \iint_{S_-} (\hat{e}_{tm} \times \hat{h}_m)\cdot\hat{z}\, e^{j2k_{zm}z}\, dS$$

$$\therefore \iint_{S^-} [(E^- \times \hat{h}_m) - (\hat{e}_m^- \times \vec{H}^-)]\cdot(-\hat{z})\, dS = 0$$

equal terms

Now Look at Integral over S^+

$$\iint_{S^+} (E^+ \times h_m^-)\cdot\hat{z}\, dS$$

$$= -C_m^+ \iint (e_{tm} \times h_{tm})\cdot dz\, dS$$

$$\iint_{S^+} (\hat{e}_m^- \times H^+)\cdot\hat{z}\, dS$$

$$= C_m^+ \iint (e_{tm} \times h_{tm})\cdot dz\, dS$$

$$C_m^+ = \frac{\iiint_V \left[(\hat{e}_{tm} - \hat{e}_{zm}) \cdot \vec{J} + (\hat{h}_{tm} - \hat{h}_{zm}) \cdot \vec{M}\right] e^{jk_{zm}z}\, dV}{2 \iint (\hat{e}_{tm} \times \hat{h}_{tm}) \cdot \hat{z}\, dS}$$