FERMILAB **ENGINEERING NOTE**

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SUBJECT: Communication Concepts

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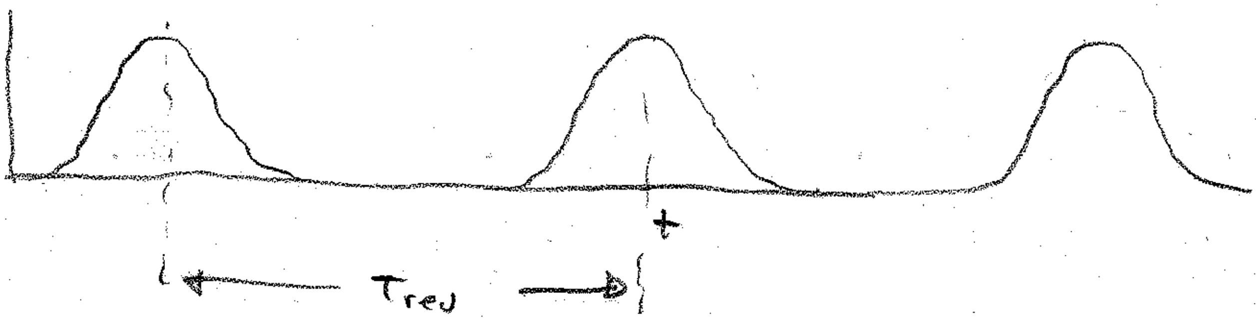
Periodic signals.

$$v(t) = \sum_{n=-\infty}^{\infty} V_r(t - nT_r)$$



Since $v(t)$ is repetative we can represent $v(t)$ as a sum of a set of orthoganal repetative functions.

We'll choose sine waves

$$e^{j 2\pi m \frac{t}{T_r}} = \cos\left(2\pi m \frac{t}{T_r}\right) + j \sin\left(2\pi m \frac{t}{T_r}\right)$$

is periodic at T_r given m is a integer

$$\omega_r = \frac{2\pi}{T_r}$$

$$v(t) = \sum_{m=-\infty}^{\infty} C_m \, e^{jm\omega_r t}$$

C_m is complex

$$v(t)\, e^{-jk\omega_r t} = \sum_{m=-\infty}^{\infty} C_m e^{j(m-k)\omega_r t}$$

$$\int_{-\frac{T_r}{2}}^{T_r/2} v(t)\, e^{-jk\omega_r t} dt = \sum_{m=-\infty}^{\infty} C_m \int_{-T/2}^{T/2} e^{j(m-k)\omega_r t}$$

$$= C_m T_r \, \delta_{m,k}$$

$$\boxed{C_m = \frac{1}{T_r} \int_{-T_r/2}^{T_r/2} v(t)\, e^{-jm\omega_r t}}$$

Note

$$C_m^* = \frac{1}{T_r} \int_{-T_r/2}^{T_r/2} v(t)^* \, e^{jm\omega_r t}$$

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If $v(t)$ is real

$$v(t)^* = v(t)$$

$$\therefore \; C_m^* = C_{-m}$$

Fourier Transforms.

Repetative signal

$$v(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_r t}$$

$$T_r C_m = \int_{-T_r/2}^{T_r/2} v(t)\, e^{-jm\omega_r t}\, dt$$

$$\omega_m = m\omega_r$$

$$\tilde{V}(\omega_m) = T_r C_m$$

$$v(t) = \sum_{m=-\infty}^{\infty} \frac{1}{T} \tilde{V}(\omega_m) e^{j\omega_m t}$$

Spacing between adjacent frequencies

$$\Delta\omega_m = m\frac{2\pi}{T_r} - (m-1)\frac{2\pi}{T_r} = \frac{2\pi}{T_r}$$

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$$v(t) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \Delta\omega_m \tilde{V}(\omega_m) e^{j\omega_m t}$$

Limit as $\Delta\omega_m \to 0$ (or $T \to \infty$)

$$\boxed{\begin{aligned} v(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(\omega) e^{j\omega t}\, d\omega \\ \tilde{V}(\omega) &= \int_{-\infty}^{\infty} v(t) e^{-j\omega t}\, dt \end{aligned}}$$

<u>Fun with Delta Functions</u>

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(\tau) e^{-j\omega\tau}\, d\tau \right) e^{j\omega t}\, d\omega$$

Interchange the order of ω, τ integration

$$v(t) = \int_{-\infty}^{\infty} v(\tau) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\, d\omega \right) d\tau$$

Definition of a δ-function

$$v(t) = \int_{-\infty}^{\infty} v(\tau)\,\delta(t-\tau)\,d\tau$$

$$\therefore \delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\,d\omega$$

or $$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(\omega-\omega')t}\,dt$$

<u>Assume $V(t)$ is periodic</u> $V(t) = V_p(t)$

Fourier Transform of a periodic series

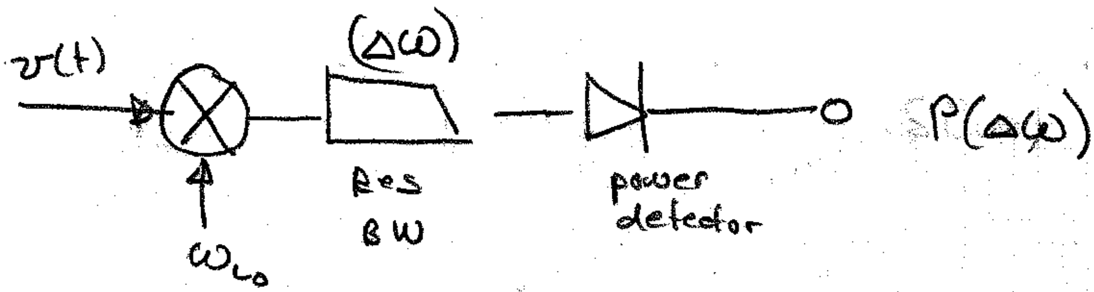
$$V_p(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_{rev}t}$$

$$\tilde{V}_p(\omega) = \int_{-\infty}^{\infty} v_p(\tau)\,e^{-j\omega\tau}\,d\tau$$

$$\tilde{V}_p(\omega) = \sum_{m=-\infty}^{\infty} C_m \int_{-\infty}^{\infty} e^{j(m\omega_{rev}-\omega)\tau}\,d\tau$$

$$\tilde{V}_p(\omega) = 2\pi \sum_{m=-\infty}^{\infty} C_m\,\delta(\omega - m\omega_{rev})$$

But spectrum analyzers do not measure voltages & currents. They measure power deposited into a filter



$$P(\Delta\omega) = S(\omega)\,\Delta\omega$$

Power Spectral density.

$$\langle p(t) \rangle = \frac{1}{2\pi}\int S(\omega)\, d\omega$$

But Time averaged power is

$$\langle p(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\,\frac{1}{R}\int_{-T/2}^{T/2} v(t)\,v(t)\,dt$$

Since $\displaystyle v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}(\omega_1)\, e^{j\omega_1 t}\, d\omega_1$

and $v(t)$ is <u>real</u>

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}^*(\omega_2)\, e^{-j\omega_2 t}\, d\omega_2$$

$$\langle p(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\,\frac{1}{R}\left(\frac{1}{2\pi}\right)^2 \int_{-T/2}^{T/2} \int_{\omega_1}\int_{\omega_2} \tilde{V}(\omega_1)\tilde{V}(\omega_2)^* e^{j(\omega_1-\omega_2)t}\, d\omega_1\, d\omega_2\, dt$$

Re-arranging integration for t & ω

$$\langle p(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\,\frac{1}{R}\left(\frac{1}{2\pi}\right)^2 \int_{\omega_1}\int_{\omega_2} \tilde{V}_1(\omega_1)\,\tilde{V}(\omega_2)^* \underbrace{\int_{-T/2}^{T/2} e^{j(\omega_1-\omega_2)t}\, dt}_{2\pi\delta \text{ function}}\; d\omega_1\, d\omega_2$$

$$\langle p(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\frac{1}{R}\int_{\omega} |V(\omega)|^2 d\omega$$

$$\boxed{S_f(\omega) = \frac{1}{R}\lim_{T\to\infty}\frac{1}{T}|V(\omega)|^2}$$

From the periodic Spectrum.

$$\tilde{V}_p(\omega) = 2\pi \sum_{m=-\infty}^{\infty} C_m\,\delta(\omega-\omega_{rev})$$

$$S_p(\omega) = \lim_{T\to\infty}\frac{1}{T}\frac{1}{R}(2\pi)^2\sum_m\sum_{m'} C_m C_{m'}^*\,\delta(\omega-m\omega_r)\,\delta(\omega-m'\omega_r)$$

Since δelta functions do not overlap for $m \neq m'$

$$S_p(\omega) = \lim_{T\to\infty}\frac{1}{T}\frac{1}{R}(2\pi)^2\sum_m |C_m|^2(\delta(\omega-m\omega_r))^2$$

Lets look at:

$$\lim_{T\to\infty} \frac{1}{T}\left(\delta(\omega - m\omega_{rev})\right)^2 = \lim_{T\to\infty} \frac{1}{T}\,\delta(\omega - m\omega_{rev})\frac{1}{2\pi}\int_{-T/2}^{T/2} e^{j(\omega - m\omega_r)}\,dt$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} e^{j(\omega - m\omega_r)}\,dt = 1 \quad \text{for } \omega = m\omega_r$$

which is constrained by the δ out front

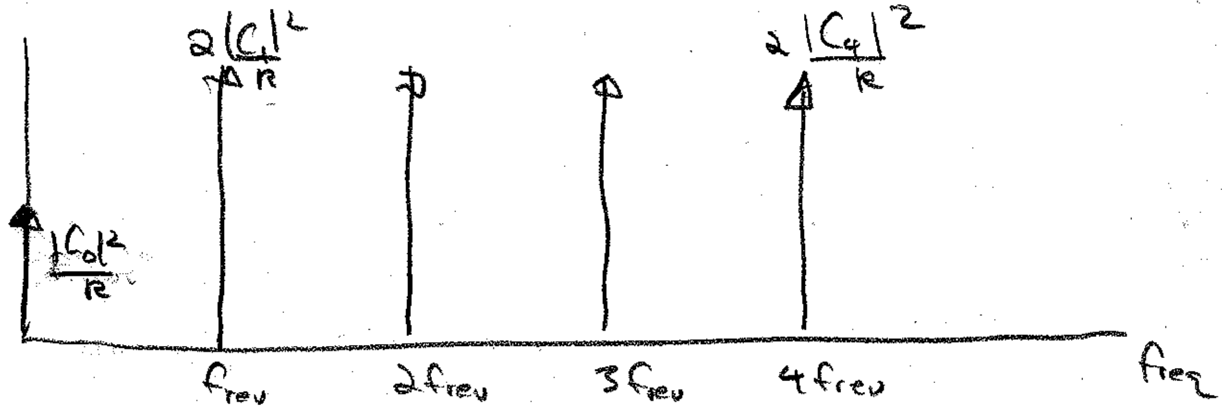
$$S_p(\omega) = \frac{2\pi}{R}\sum_{m=-\infty}^{\infty} |C_m|^2\,\delta(\omega - m\omega_{rev})$$

Since $\omega = 2\pi f$

$$S_p(f) = \frac{1}{R}\sum_{m=-\infty}^{\infty} |C_m|^2\,\delta(f - nf_{rev})$$

Also spectrum analyers don't measure phase, so they can't distinguish between negative frequencies

$$S_p(f) = \frac{|C_0|^2}{R}\delta(0) + 2\sum_{n=1}^{\infty} \frac{|C_n|^2}{R}\delta(f - n f_{rev})$$



Note that a spectrum analyzer does not measure the power spectral density. It measures the power deposited in a filter of width Δf at a frequency f

So for a periodic signal, as the resolution band width is changed, the peak signal on a spectrum analyzer does not vary

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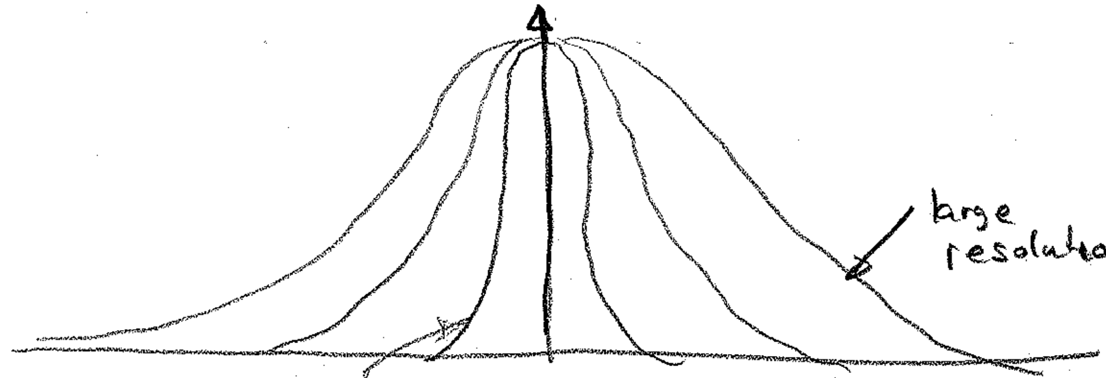

AM modulation.

Why modulate?

The size of radiation structures are proportional to the wavelength.

i.e. It is best to design antenna's at high frequencies.

Information bandwidth can be constrained

(Our ears hear 20Hz-20 kHz

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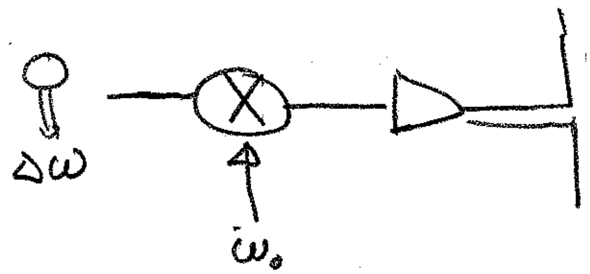


Usually $\Delta\omega \ll \omega_0$

AM modulation

Information $V_{info} = m \cos \omega_s t$

volume (modulation index)

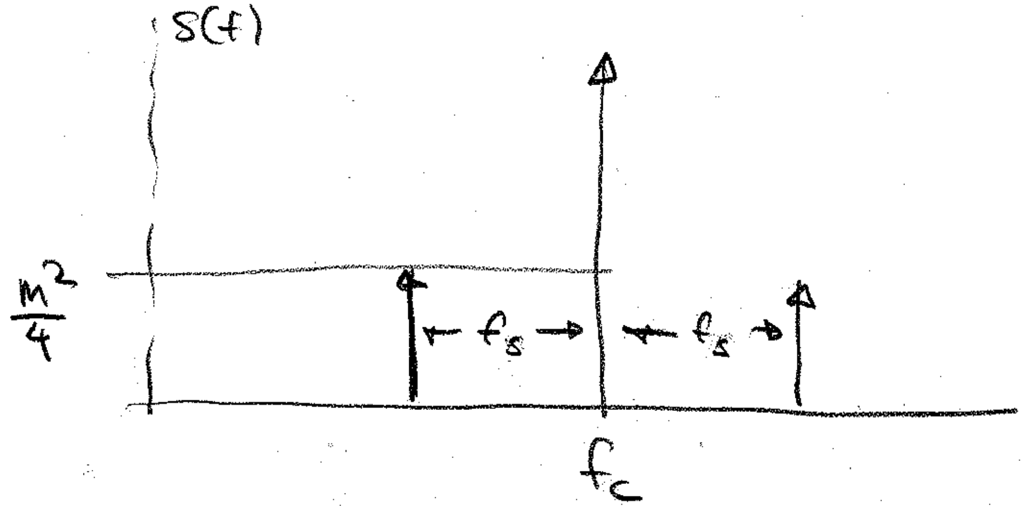
pitch (modulation frequency)

Modulated signal

$$V(t) = V_0 \left(1 + m \cos(\omega_s t)\right) \cos \omega_c t$$

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

$$V(t) = V_0 \cos(\omega_c t) + \frac{m V_0}{2} \cos\left((\omega_c + \omega_s)t\right) + \frac{m V_0}{2} \cos\left((\omega_c - \omega_s)t\right)$$



FM modulation.

$$\omega = \omega_c + \omega_m \cos(\omega_s t)$$

ω_s is the modulation frequency

ω_c is the carrier frequency

ω_m is the modulation amplitude

Can't really multiply by t and put any a cosine around it

Frequency is better defined as the derivative of phase

$$\omega = \frac{d\phi}{dt}$$

$$\phi = \int \omega(t)\, dt$$

$$\phi = \omega_c t + \frac{\omega_m}{\omega_s} \sin(\omega_s t)$$

$$V(t) = V_0 \cos\left(\omega_c t + \frac{\omega_m}{\omega_s} \sin \omega_s t\right)$$

$$V(t) = V_0 \,\mathrm{Re}\left(e^{j\left(\omega_c t + \frac{\omega_m}{\omega_s} \sin \omega_s t\right)}\right)$$

$$V_c(t) = V_0 e^{j\left(\omega_c t + \frac{\omega_m}{\omega_s} \sin \omega_s t\right)}$$

$$V_c(t) = V_0 e^{j\omega_c t} e^{j\left(\frac{\omega_m}{\omega_s} \sin \omega_s t\right)}$$

$$e^{jz\sin(x)} = \sum_{n=-\infty}^{\infty} J_n(z) e^{jnx}$$

$$V_c(t) = V_0 e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n\left(\frac{\omega_m}{\omega_s}\right) e^{jn\omega_s t}$$

$$V_c(t) = V_0 \sum_{n=-\infty}^{\infty} J_n\left(\frac{\omega_m}{\omega_s}\right) e^{j(\omega_c + n\omega_s)t}$$

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$$v(t) = V_0 \sum_{n=-\infty}^{\infty} J_n\left(\frac{\omega_m}{\omega_s}\right) \cos\left((\omega_c + n\omega_s)t\right)$$

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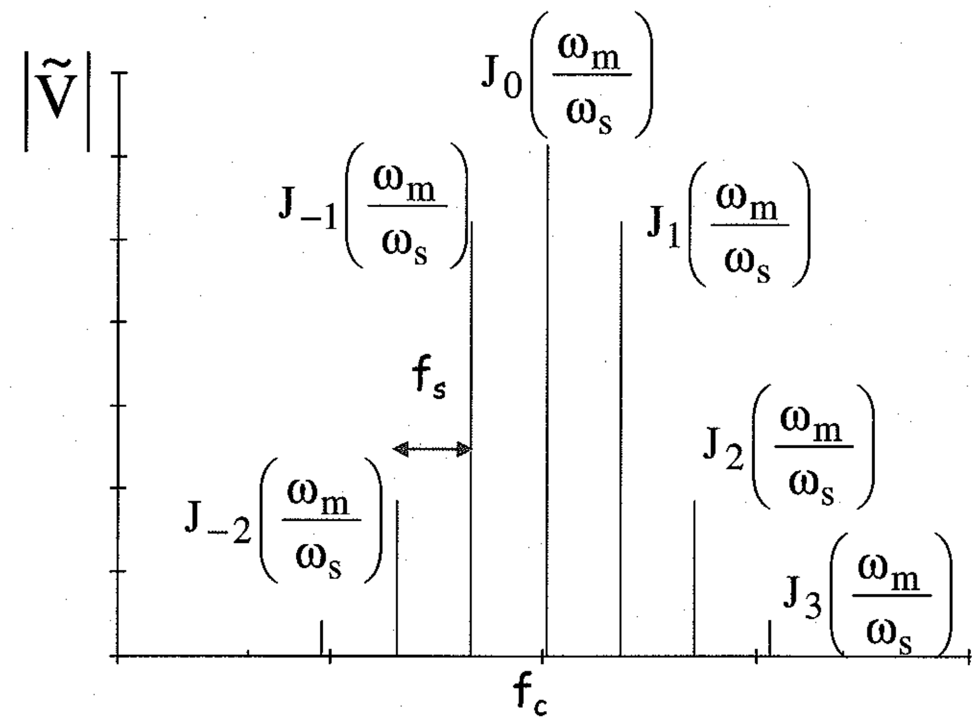
Frequency Modulation

$$v(t) = \cos\left(\omega_c t + \frac{\omega_m}{\omega_s}\sin(\omega_s t)\right)$$

$$v(t) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{\omega_m}{\omega_s}\right)\cos((\omega_c + m\omega_s)t)$$



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Bessel Function Magic

The complex exponential of a sine function can be "simplified" by using Bessel functions

$$e^{jz\sin(x)} = \sum_{m=-\infty}^{\infty} J_m(z) e^{jmx}$$