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ENGINEERING NOTE

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SUBJECT: Laplace Transforms

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Fourier Transform Pair

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{j\omega t} d\omega$$

ω is a real number.

The problem with Fourier transforms is that they cannot handle intial conditions due to the fact $\int_{-\infty}^{\infty}$

To handle initial conditions, consider the Laplace transform

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_{0}^{\infty} f(t) e^{-st} dt$$

s is a complex number!

$$s = \sigma + j\omega$$

Since s is complex number, the inverse transform is hard to compute directly

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$$\mathcal{L}^{-1}[\tilde{f}(s)] = f(t)$$

Common transform pairs

$f(t)$	$\tilde{f}(s)$
$\delta(t)$	1
$u(t)$ (step)	$1/s$
t	$1/s^2$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t} \cos \omega t$	$\frac{s+a}{(s+\alpha)^2+\omega^2}$

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$f(t)$	$\tilde{f}(s)$
$\dfrac{df}{dt}$	$s\tilde{f}(s) - f(0)$
$\dfrac{d^2f}{dt^2}$	$s^2\tilde{f}(s) - sf(0) - \left.\dfrac{df}{dt}\right\rvert_{t=0}$
$\int_0^t f(t)\,dt$	$\dfrac{\tilde{f}(s)}{s} + \dfrac{1}{s}\left.\left[\int_0^t f(t)\,dt\right]\right\rvert_{t=0}$
$f(t-\tau)u(t-\tau)$	$e^{-s\tau}\tilde{f}(s)$

Inital Value Theorem

$$\lim_{t\to 0^+} = f(0^+) = \lim_{s\to\infty} s f(s)$$

$$f(t) = e^{-\alpha t}$$

$$\tilde{f}(s) = \frac{1}{s+\alpha}$$

$$\lim_{s\to\infty} s f(s) = \lim_{s\to\infty} \frac{s}{s+\alpha} = 1$$

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Final value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s f(s)$$

(only is $f(t)$ is stable)

Uncharged Capacitor

$$i(t) = C \frac{dv}{dt}$$

$$\tilde{i}(s) = sC\,\tilde{v}(s) - C\,\underbrace{v(0)}_{0}$$

Uncharged Inductor.

$$v(t) = L \frac{di}{dt}$$

$$\tilde{v}(s) = sL\,\tilde{i}(s)$$

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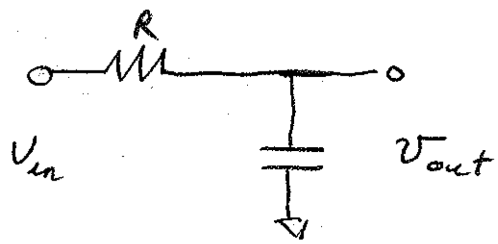
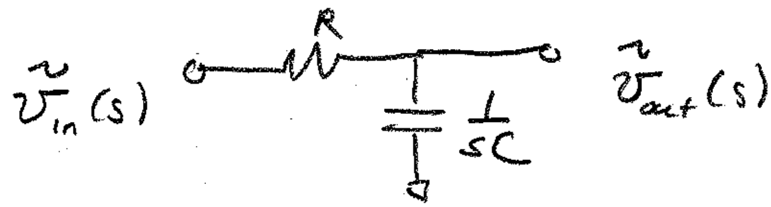
Low Pass Filter



In Laplace Domain (with everything uncharged)



<u>Voltage divider</u>

$$\tilde{v}_{out}(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \tilde{v}_{in}(s)$$

$$= \frac{1}{1 + sRC}$$

What is the frequency response

$$s \Rightarrow j\omega$$

$$\tilde{V}_{out}(j\omega) = \frac{1}{1 + j\omega RC} \tilde{v}_{in}(j\omega)$$

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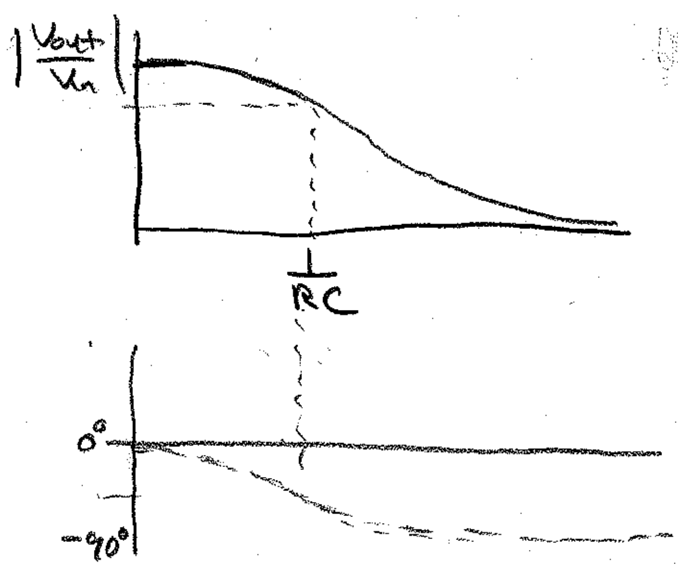
When $\omega = \frac{1}{RC}$

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+1j} = \frac{1-1j}{2}$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{2}}$$

$$\angle\left(\frac{V_{out}}{V_{in}}\right) = -45^\circ$$



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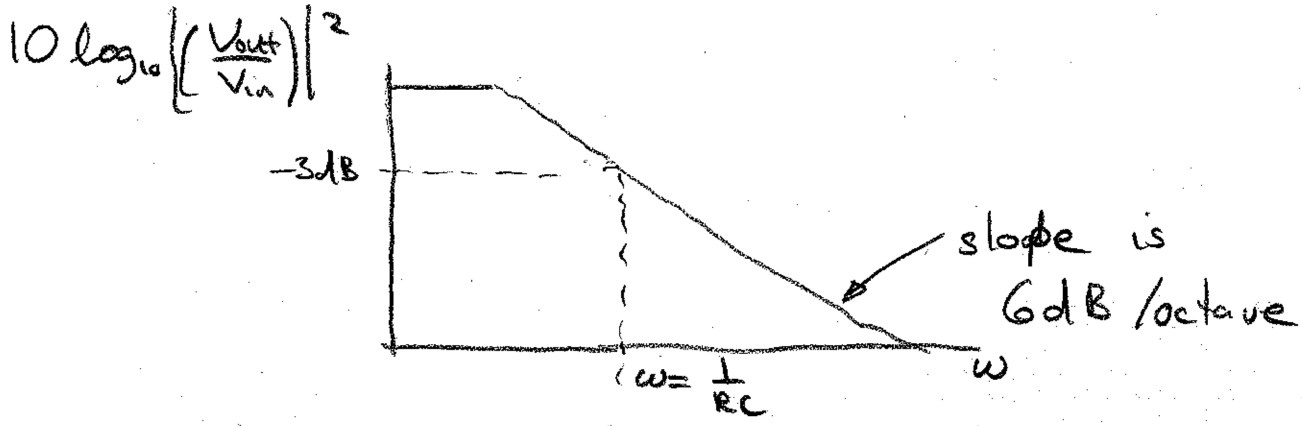
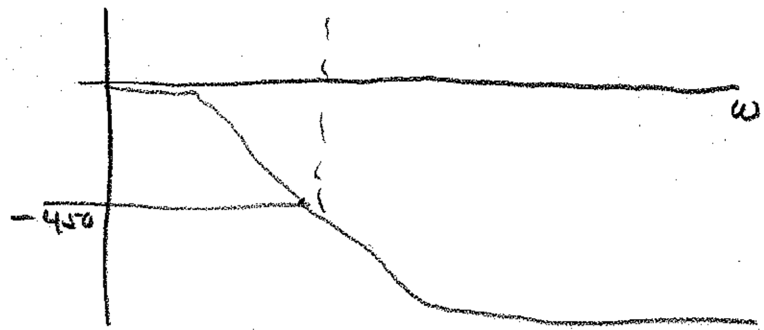




The 3dB bandwidth

$$\Delta\omega = \frac{1}{RC}$$

$$\Delta f = \frac{1}{2\pi RC}$$

What does bandwidth mean?

$$\tilde{V}_{out}(s) = \frac{1}{1+sRC}\tilde{V}_{in}(s)$$

Let $V_{in}(t) = v_0 \Delta\tau \; \delta(t)$ (get units right)

1/time

$$\tilde{V}_{in}(s) = v_0 \Delta\tau$$

$$\tilde{v}_{out}(s) = \frac{v_0 \Delta\tau}{1+sRC}$$

$$\tilde{v}_{out}(s) = \frac{v_0 \Delta\tau}{RC}\frac{1}{\left(s+\frac{1}{RC}\right)}$$

Let $\Delta\tau = RC$

$$\tilde{v}_{out}(s) = \tilde{v}_0 \frac{1}{\left(s+\frac{1}{\Delta\tau}\right)}$$

$$v_{out}(t) = v_0 \, e^{-t/\Delta\tau}$$

Now consider a second impulse on the circuit.

$$v_{in_2}(t) = v_0 \Delta\tau \, \delta(t - \tau_d) u(t - \tau_d)$$

$$\tilde{v}_{in_2}(s) = v_0 \Delta\tau \, e^{-s\tau_d}$$

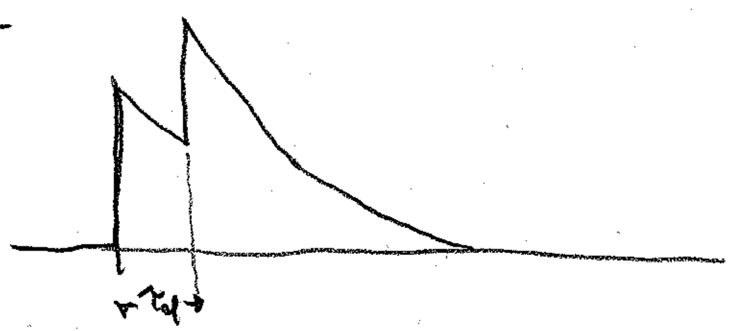
$$\tilde{v}_{out_2}(s) = v_0 \frac{e^{-s\tau_d}}{(s + 1/\Delta\tau)}$$

$$v_{out_2}(t) = v_0 \, e^{-(t-\tau_d)/\Delta\tau} \, u(t - \tau_d)$$

Now consider the two impulses together

$$v_{in_{1+2}}(t) = v_0 \Delta\tau \, \delta(t) + v_0 \Delta\tau \, \delta(t - \tau_d) \, u(t - \tau_d)$$

$$v_{out_{1+2}}(t) = v_0 e^{-t/\Delta\tau} + v_0 e^{-(t-\tau_d)/\Delta\tau} u(t - \tau_d)$$



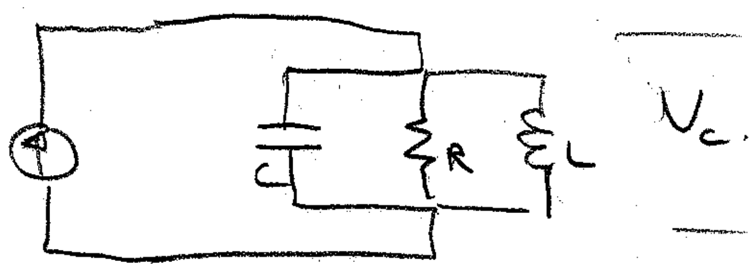
if $\tau_d \ll \Delta\tau$ we cannot distinguish the two impulses on the output

$\Delta\tau$ is a measure of our ability to resolve the two impulses

$$\Delta\tau = \frac{1}{2\pi\,\Delta f}$$

$\therefore$ Bandwidth is a measure of how well a circuit can resolve impulses

<u>Consider the cavity circuit</u>



$$Z(s) = \frac{1}{sC + \frac{1}{R} + \frac{1}{sL}}$$

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$$Z(s) = \frac{s/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Define $\omega_0^2 = \frac{1}{LC}$

$$Q = \omega_0 RC$$

$$Z(s) = \frac{s\,\omega_0 R/Q}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

$$i_{gen}(t) = \underset{\text{charge}}{q}\,\delta(t)$$

Remember

$$\mathcal{L}(e^{-\alpha t}\sin\omega_r t) = \frac{\omega_r}{(s+\alpha)^2 + \omega_r^2}$$

$$\mathcal{L}(e^{-\alpha t}\cos\omega_r t) = \frac{s+\alpha}{(s+\alpha)^2 + \omega_r^2}$$

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$$\mathcal{L}\left(e^{-\alpha t}\cos\omega_r t - \frac{\alpha}{\omega_r} e^{-\alpha t}\sin\omega t\right) = \frac{s}{(s+\alpha)^2+\omega_r^2}$$

Comparing

$$2\alpha = \frac{\omega_0}{Q}$$

$$\alpha = \frac{1}{2}\frac{\omega_0}{Q}$$

$$\omega_0^2 = \alpha^2 + \omega_r^2$$

$$\omega_r^2 = \omega_0^2\left(1 - \frac{1}{4Q^2}\right)$$

$$v(t) = q\omega_0 \frac{R}{Q}\, e^{-\frac{\omega_0}{2Q}t}$$

$$\cdot\left[\cos\omega_r t - \frac{1}{\sqrt{4Q^2-1}}\sin\omega_r t\right]$$

<u>Note</u>

1) Cavity rings down at $\tau = \frac{2Q}{\omega_0}$

2) Ring frequency slighty shifted down from resonant freq.

3) Amplitude $\propto R/Q$ not R!

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Look at

$$\mathcal{L}(e^{-\alpha t}\sin\omega_r t) = \frac{\omega_r}{(s+\alpha)^2+\omega_r^2}$$

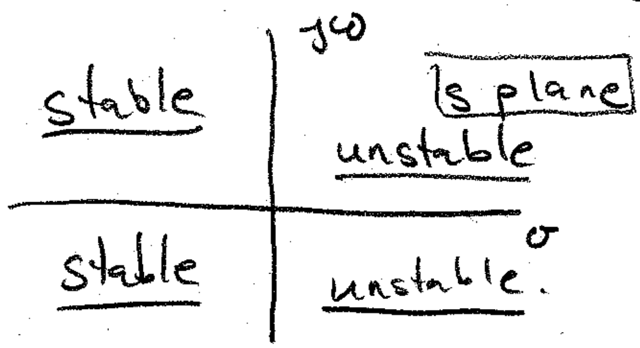
$$(s+\alpha)^2+\omega_r^2 = (s-(j\omega_r-\alpha))(s-(-j\omega_r-\alpha))$$

poles of the response

(places where denominator goes to zero)

The characteristic frequencies of the system is determined by the poles of the Laplace Transform.

Note that for the system to be stable, the real part of the poles must be < 0. (i.e., the poles must be on the left hand side of the splane



$s = \sigma + j\omega$

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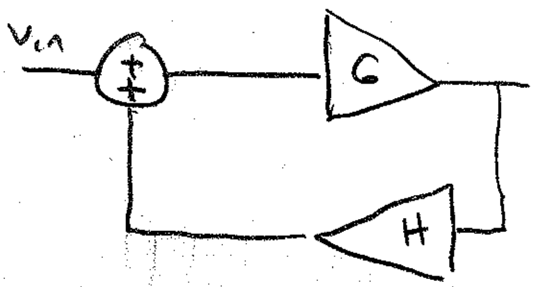
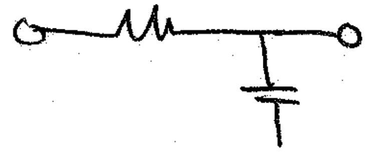
Consider a Feedback system



$H =$ 

$$V_{out} = G\,V_{in} + G\,H\,V_{out}$$

$$V_{out} = \frac{G}{1 - GH}\,V_{out}$$

let $H = \dfrac{1}{1 + s\tau_f}$ (low pass filter)

$$V_{out} = \frac{G(1 + s\tau_f)}{1 + s\tau_f - G}$$

$$= \frac{G(1 + s\tau_f)}{s + \dfrac{1-G}{\tau_f}}$$

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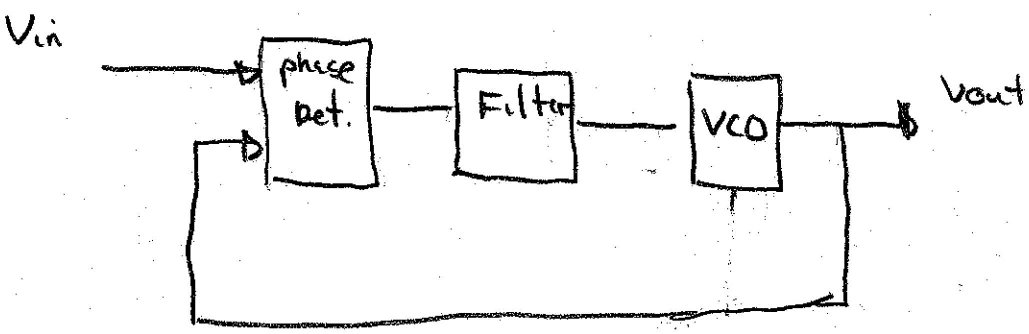
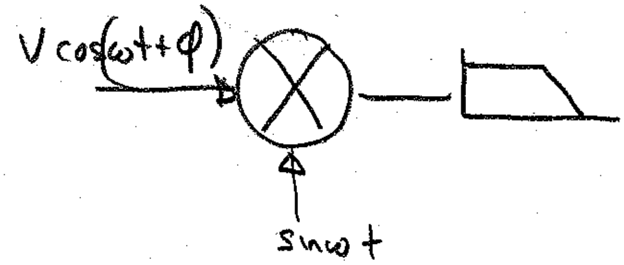
if $G > 1$ system will be unstable!

Phase Lock Loop



How do you make a phase detector?

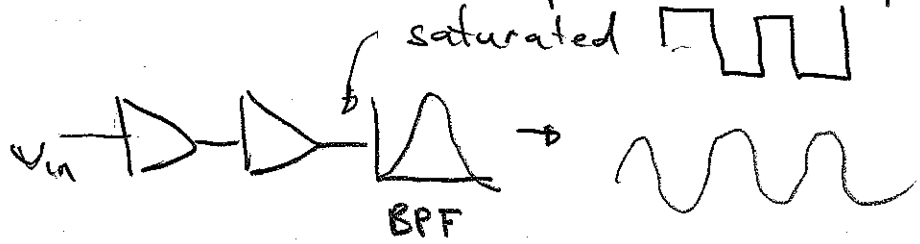


$$V_{mixer} = V\cos(\omega t+\varphi)\sin\omega t$$

$$= \frac{1}{2} V \sin(2\omega t+\varphi) + \frac{V}{2}\sin(\varphi)$$

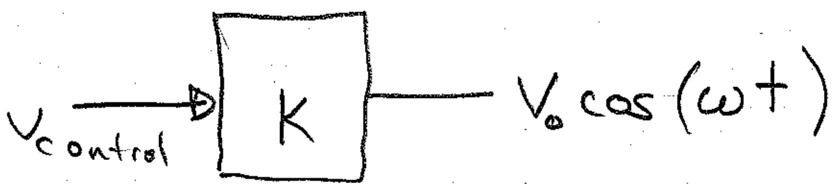
$$V_{filter} = \frac{V}{2}\sin\phi$$

To remove Amplitude dependence



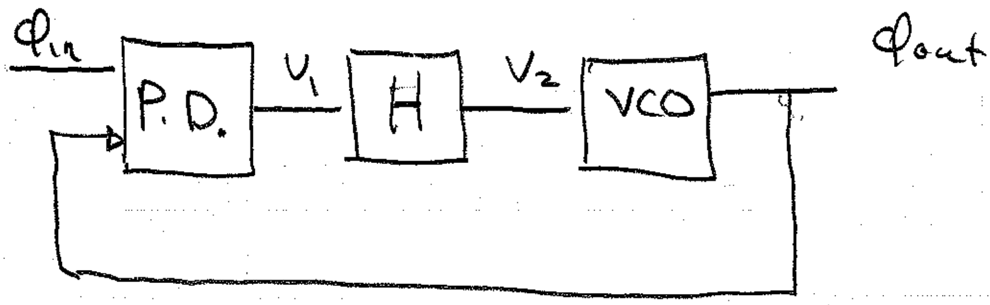
A VCO is a voltage controlled oscillator

The frequency out of the oscillator is proportional to the controll voltage



$$\omega = 2\pi K V_c$$

Lets use the phase as our variable



$$V_1 = \varphi_{in} - \varphi_{out}$$

$$v_2 = H(s)\, v_1$$

$$\varphi_{out} = \int 2\pi f_{out}\, dt$$

$$f_{out} = K V_2$$

$$\varphi_{out} = \frac{2\pi K}{s} V_2$$

$$\varphi_{out} = \frac{2\pi K}{s} H(s) \left(\varphi_{in} - \varphi_{out} \right)$$

$$= 2\pi K \frac{H}{s} \varphi_{in} - 2\pi K \frac{H}{s} \varphi_o$$

$$\varphi_{out} = \frac{2\pi K \frac{H}{s}}{1 + 2\pi K \frac{H}{s}} \varphi_{in}$$

Let $H(s) = \dfrac{1}{1 + s\tau_h}$

$$G = 2\pi K \quad {}^{1/\mathrm{Hz}}$$

$$\varphi_o = \frac{G \quad 1}{s(1+s\tau_h)\left(1 + \frac{G \quad 1}{s(1+s\tau_h)}\right)}$$

$$= \frac{G}{s^2 \tau_h + s + G}$$

$$\varphi_o = \frac{G/\tau_h}{s^2 + \frac{s}{\tau_h} + \frac{G}{\tau_h}} \varphi_{in}$$

$$S_p = \frac{-1}{2\tau_h} \pm \left(\left(\frac{1}{2\tau_h}\right)^2 - \frac{G}{\tau_h} \right)^{1/2}$$

Critical damping

$$G = \frac{1}{4\tau_h}$$

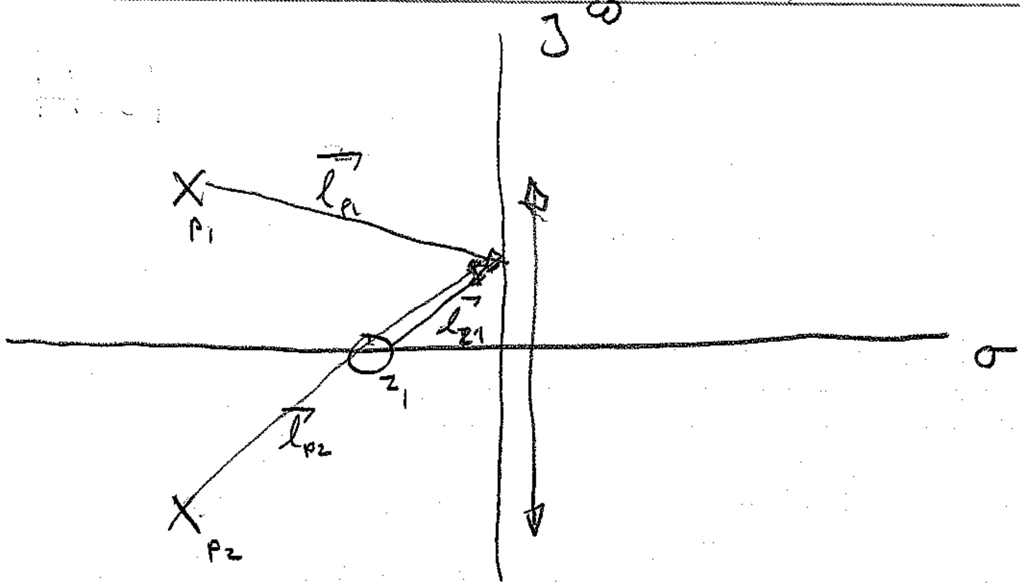
$$K = \frac{1}{8\pi} \frac{1}{\tau_h}$$

$$\varphi_o = \frac{\left(1/2\tau_h\right)^2}{\left(s + \frac{1}{2\tau_h}\right)^2} \varphi_{in}$$

<u>Plotting the frequency response</u>

In General

$$H(s) = \frac{(s - s_{z_1})(s - s_{z_2}) \ldots (s - s_{z_{n-1}})}{(s - s_{p_1})(s - s_{p_2}) \ldots (s - s_{p_n})}$$

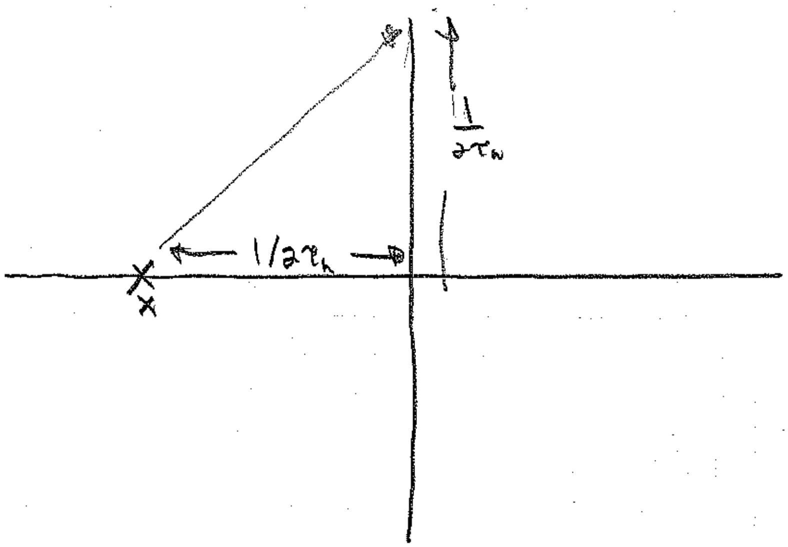


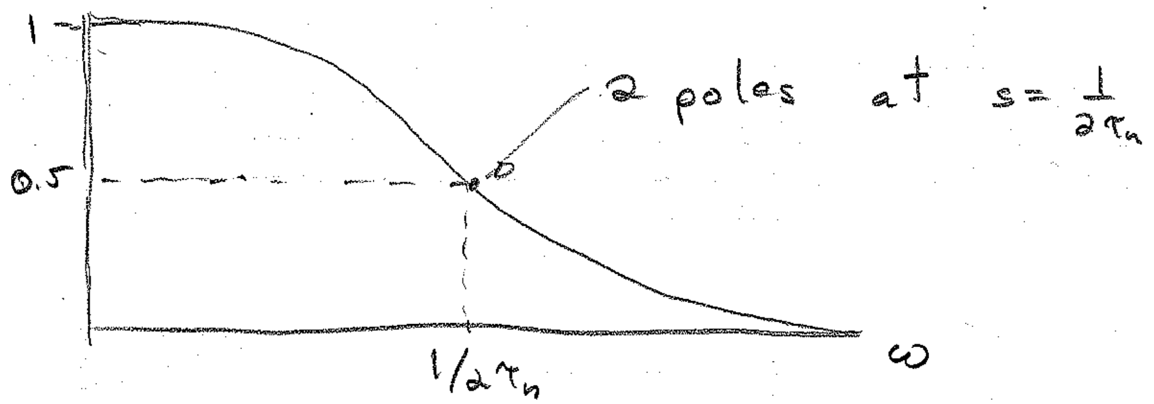
$$|H(j\omega)| = \frac{\prod |\vec{l}_{z_i}|}{\prod |\vec{l}_{p_i}|}$$

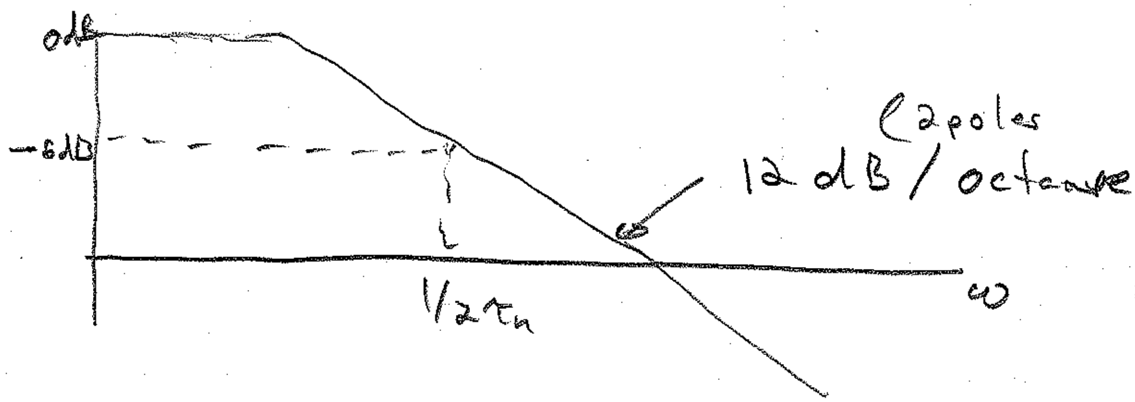
$$\angle H(j\omega) = \sum \angle \vec{l}_{z_i} - \sum \angle \vec{l}_{p_i}$$

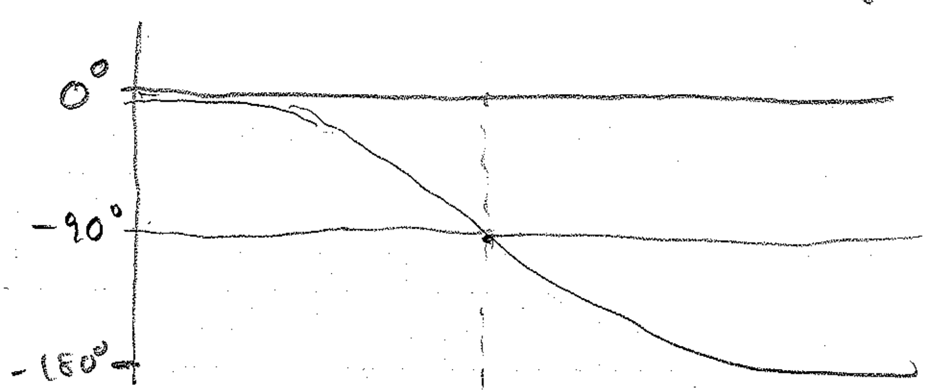
Look at phase lock loop response

$$\phi_{out}(s) = \frac{(1/2\tau_n)^2}{\left(s + \frac{1}{2\tau_n}\right)^2} \phi_{in}$$


1/2τn
1/2τn
x


1
0.5
2 poles at s = 1/2τn
ω
1/2τn


0dB
-6dB
(2 poles
12 dB / octave
ω
1/2τn


0°
-90°
-180°

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$$V_2 = \frac{s}{G} \varphi_{out}$$

For critical damping

$$G = \frac{1}{4\tau_n}$$

$$V_2 = s\, 4\tau_n\, \varphi_{out}$$

$$V_2 = \frac{s(1/2\tau_n)^2\, 4\tau_n}{\left(s + \frac{1}{2\tau_n}\right)^2} \varphi_{in}$$

$$= \frac{s/\tau_n}{\left(s + \frac{1}{2\tau_n}\right)^2} \varphi_{in}$$